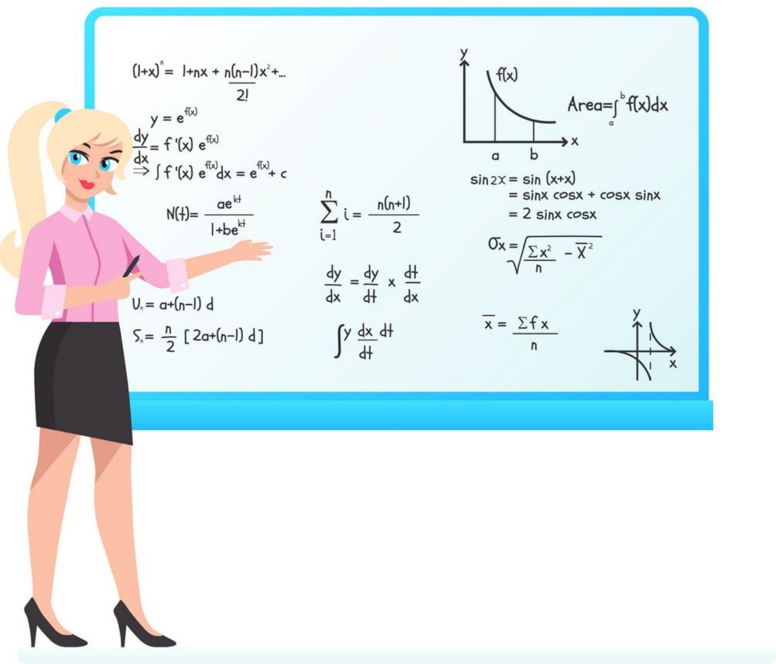
My Maths Cloud

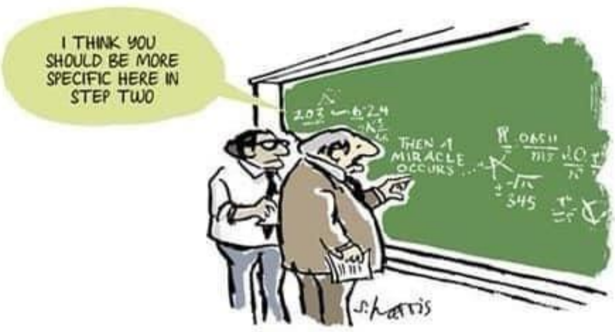
Constructing Proofs

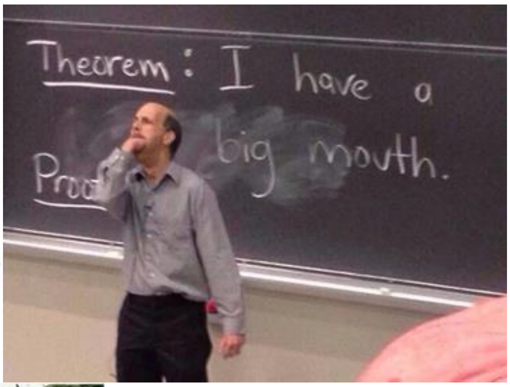


www.mymathscloud.com
francesca@mymathscloud.com

Proofs:

Table of Contents





1 Advice

Getting good at proofs is tough honestly - one of those things that just takes a lot of practice. Especially because there's often a lot of scrap work done behind a proof that isn't shown, and plenty of trial and error. Everyone you ask will recommend practicing and a lot of it. But there is one problem with the "practise, practise, practise" mantra. Practise what? Where are the lists of similar-but-not-quite-identical things to prove to practise on? One can find lists of integrals to do and lists of matrices to solve, but it's hard coming up with lists of things to prove. Of course, practise is correct, but just as with anything else in mathematics, there are guidelines to help get you started. The first thing to realise is that reading other people's proofs is not guaranteed to give you any insight as to how the proof was developed. A proof is meant to convince someone of a result, so a proof points to the theorem (or whatever) and knowing how the proof was constructed does not (or at least, should not) lend any extra weight to our confidence in the theorem. Proofs can certainly be written in such a way to show how they were constructed, and when teaching lecturers should make sure to present some proofs in this way, but to do it every time would be tedious. Therefore, many academic texts are simply providing what is necessary to deem the proof as correct, being concise at the expense of being informative. If time allowed, I'm sure lecturers would try to present all proofs in such a way so that a mathematician might discover the proof, with investigation that results in a useful discovery. 3Blue1Brown on YouTube is the closest it gets to this style.

Even with all the above in place, sometimes it really does take a lot of mental effort, pondering and coming back to something to really understand what's going on. Finding proofs where the reasoning is well explained, coupled with already having a good understanding of the concepts involved in the proof will help immensely. Proofs are a hard skill to teach. The key to becoming good at writing proofs is to really understand what you are being asked/trying to prove, which is harder than it may seem. So, having said all this, what are some tips for constructing a proof?

Before you even start, write out clearly what must be proven. In particular, be aware of whether the statement's proof requires universal generalization, negation, etc. Secondly, make sure you are very familiar with the usual proving techniques such as proof by contradiction, contrapositive proof, proof by construction, direct proof, equivalence proof etc. Make yourself acquainted with the premises. How can the statement fail if a single premise is left out? If you can't prove a theorem using any of the usual proving methods, a "proof directly from the definitions" usually does the trick. This is why it is vital to you learn (and understand!) the definitions of the terms in the theorem. Write down all definitions and theorems that you feel might be relevant to the topic at hand. Often, I have seen students hampered by not really knowing all of the definitions in the problem statement. Next, try to understand what the theorem actually means (this is the harder bit!). In the proof, check you have exhausted all the assumptions (no theorem will have more assumptions than needed!). Another good strategy is to work with specific examples until you understand the problem. Plug in numbers and see why the theorem seems to be true. Also, try to construct a counterexample. The reason counterexamples fail often leads to a way to prove the statement.
So, a nice starting point to have is:

- Find yourself a specific numerical example of the problem statement and check the conditions. Maybe you note a way how the premises enforce the validity of the statement for this example.
- Try to find a counterexample. You (probably) won't find one, but you might notice what kind of obstacles prevent you from finding it.
- Check extremes. If the statement says "for all real numbers with $0< r <2$ ...", then check what would happen with $r = 0$ and $r = 2$.

Another technique that I think is very useful for people first learning how to prove things is: If you are trying to "prove statement X," take the point of view that you are unsure if statement X is true. Then, try to decide if it is true or not. Ad already mentioned, seek counterexamples. Seek evidence that might suggest X is true as well. If at some point you become convinced that X is actually false, great! Try to convince somebody else. If, on the other hand, you become convinced that X is true, great! However, you became convinced can be the basis of your proof. You need your proof-writing skills to be linked to the process by which you come to believe what's true and what isn't. Learning how to prove is nothing more than learning how to write down an absolutely convincing argument. Math has developed a lot of techniques, tricks, common argument patterns, etc., giving the impression that there is a whole body of stuff one has to master, but at its heart, a proof is nothing more than a logical argument that serves to convince everybody that something is true. To learn how to make good arguments, you need to be tuned into what is convincing and what isn't, and the authentic way to do this is to stay tuned in to what convinces *you* and what doesn't. So, in trying to create a proof, the best thing is to take the point of view that you aren't sure if it is even true, and actually decide for yourself if and why you think it is, being as sceptical as you possibly can. If you become convinced it is true, no matter how sceptical you try to be, then whatever convinced you can be turned into your proof. As an aside, I believe that those of us who are experienced at writing proofs have all, at least on some (conscious or unconscious) level, developed this habit of taking the point of view that we are not sure if it is true. Then we write the proof to convince ourselves.

While many will ask for proof methods/techniques, what they also really need are proof-finding strategies. This is a large field! Here are just a few hints:

- If you get stuck, reach out to the lecturer and take full advantage of their office hours. They're usually happy to help if you're taking initiative. Don't be afraid to ask them to critique your proofs. If you are able to identify at home where you are stuck on something, it is invaluable to then be able to go and tell them this as they will probably be able to provide you that missing insight that they didn't realise they should provide in their lecture notes or clarify something that you missed or wasn't completely clear to you. Many times, a written proof omits details that the author thought were unnecessary or even trivial due to the assumed prerequisite knowledge of the reader. Unfortunately lecturers are prone to making the same mistake despite dealing with students constantly; they don't know what you don't know until you tell them.

- Rewrite your lecture notes and proofs of statements so that you understand the steps and the logic. Go through all the textbook exercises, even if you're only assigned a few. Ask permission to record the lecture so you can stop taking active notes in class. Read the text before the lecture if there is one, and during class simply copy whatever was on the board on scrap paper if you need. Then go home to your "actual notebook" and work through the whole lecture again, pausing where you need to and write down additional information where necessary based on what was said. This will allow you to spend more time focusing during class, so you can follow along and ask questions, as well as gave you an opportunity to review well after class. Very time consuming, but you WILL see a difference in your understanding of the material.

- Math is not a spectator sport - you have to do it to make sure you know how. Keep going and analysing your mistakes. The more wrong turns that you take, the better the lessons you will be taught. You get better at proofs the same way you get better at basketball or carpentry: lots and lots of practice. (In particular, like in basketball and carpentry, you can only get so far by just reading books.) Of course, there's good practice and bad practice, so be careful.

- Work with others. Do not underestimate strength in numbers; when you are just one person trying to understand and tackle a problem or subject, it is hard. Even a group of two or three will do much better than one alone. Look at what someone else has done in a proof and ask questions. Ask how they came up with the idea and ask that person to explain the proof to you. Also, do the same for them. Explaining your proofs to a classmate and the reflection upon having them ask you questions is important, so don't be afraid to learn by example - seeing how other people write the proofs can be a huge help. Don't get disheartened if others are better than you. For some reason some people are just better than others at proving things, and I'm not saying that they're just smarter, it's only that the thought processes involved just seem to come more naturally to some persons. Don't let these types of people put you off!

- Do not give up!! Try everything. Students often get stuck on proofs because they try one idea that does not work and give up. Often one goes through several bad ideas before getting anywhere on a proof.

- Get the textbooks recommended in class and review the proofs. First read your textbook and try to prove all the important theorems as you encounter them in the text. Do not memorise proofs of theorems, it will not help you in the exam. Learn to look at a theorem and see if you can figure out a proof approach.

- For writing proofs specific to a course (e.g. real analysis) you'll generally find that the same basic ideas are being used over and over again, and once you learn to recognize when these ideas will be useful your life will be much easier. For real analysis in particular, become very very familiar with epsilon delta proofs and triangle inequality.

2 Textbooks

It doesn't help that many proofs in textbooks are written in a style that makes it nearly impossible to see how someone could have come up with the proof from first principles. This is an unfortunate tendency, and you should find a different textbook if this is too much of an issue. (Alternately, you should see if there are better proofs online, for example on someone's blog. Tim Gowers and Terence Tao are fond of writing up conceptual proofs of things, and more generally their blogs are a great source of insight into how mathematicians think). Once you build up a little problem-solving skill, another way to fix this is to reprove things yourself. It helps if you can't remember what the proof in the textbook is.

There are some books (especially the first 5 below) that specifically address the topic of introducing students to the task of proving things. What they usually do is to begin by explaining some basic logic and them they build up some easy facts from set theory, binary relations and functions and in the way they introduce some proof techniques, such as "proof by contradiction", "proof by contrapositive", etc. The topic of learning proofs is generally called "Math Reasoning" or "Discrete Math" in the textbook world.

Top 5 books:

- **How To Solve it, Polya**
  This book is a classic. Every starting undergraduate should read this book. Unlike many other books I've seen, this actually does contain guidance on how to construct a proof "out of nothing". Polya really talks about problem solving and how to think about mathematical problems. He also talks a little about heuristics, and tricks of the trade, so to speak.

- **How To Read And Do Proofs – An Intro to Mathematical Thought Processes, Solow**
  This is often a favourite of many students. Not only is it clear and concise, but it has really wonderful examples from all over mathematics.

- **How To Prove It, Velleman**

- **Proofs and Fundamentals, A First Course In Abstract Mathematics, Bloch**

- **Book Of Proof, Hammock**
  This book has some great recommendations and suggestions about proof writing.

Other books:

- **Mathematical Proofs: A Transition To Mathematical Proofs, Chartrand**
  The presentation for this book is friendly and contains a lot of worked examples.

- **Thinking Mathematically, Mason**

- **Mathematical Reasoning, Sundstrom**

- **Proof Patterns, Joshi**

- **Bridge to Abstract Mathematics, Morash**

- **Discrete Mathematics, Reasoning and Proof with Puzzles Patterns and Games, Ensley & Crawley**

- **Discrete Mathematics with Proof, Gossett**

- **How To Think Like A Mathematician, Houston**

- **Write Your Own Proofs, Babich**

- **Reading, Writing And Proving, Daepp & Gorkin**

- **A Logical Introduction To Proof, Cunningham**

- **The Nuts and Bolts Of Proofs, Cupillari**

- **Writing Proofs in Analysis, Kane**

- **Analysis with Introduction to proofs**
The first chapter focuses on logic and proofs and many students find it quite useful. Of course to master the art of proofs you have to keep practicing, but knowledge of some basics such that instead of proving the statement "A implies B you can prove contrapositive "not B implies not A", and how it differs from "Proof by contradiction", and the fact that negation of "A implies B" is "A and not B", and how to deal with existential and universal quantifiers, and so forth and so forth are important. Those are useful techniques one should master. Incidentally, as an introduction to analysis, the aforementioned book is quite mediocre. So, this is only the chapter on logic and proof which is worth reading.

- **The Art and Craft Of Problem Solving, Zeitz**

- **Problem Solving Strategies,** Engel **(somewhat high-level)**

- **Proofs That Really Count, Benjamin and Quinn**

- **The Art of Proof, Beck**

The problem books below are useful references for working problems and proofs

- Schaum's Outline of Discrete Mathematics
- Schaum's 2000 Solved Problems in Discrete Mathematics, Lipschutz
- Concrete Mathematics: A Foundation for Computer Science, Graham
- Finite and Discrete Math Problem Solver, Lutfiyya

Extra pdfs:

- There are some notes called "Reading, discovering and writing proofs" from the University of Waterloo.
https://cs.uwaterloo.ca/~cbruni/pdfs/Math135SeptDec2015/RDW_1.pdf
These notes teach you how to write a proof, taking as examples some elementary number theory. You can find these notes online.

- Discrete Math
http://www.cs.dartmouth.edu/~ac/Teach/CS19-Winter06/SlidesAndNotes/lec12induction.pdf

- Introduction To Proof, Taylor
https://www.researchgate.net/publication/319377140_A_TeXas_Style_Introduction_to_Proof

3 Intuition Videos

- 3Blue1Brown

https://www.youtube.com/channel/UCYO_jab_esuFRV4b17AJtAw
This is great for conceptual understanding

- TheMathSorcerer

https://www.youtube.com/user/themathsorcerer

- Blackpenredpen

https://www.youtube.com/user/blackpenredpen
This guy is awesome and so motivating. Check out his 100 derivatives and 100 series videos ☺

- Michael Penn

https://www.youtube.com/channel/UC6jM0RFkr4eSkzT5Gx0HOAw

- Dr Peyam

https://www.youtube.com/channel/UCoOjTxz-u5zU0W38zMkQIFw

- Professor Dave

https://www.youtube.com/channel/UC0cd_-e49hZpWLH3UIwoWRA

- Insights into Mathematics

https://www.youtube.com/user/njwildberger