IYGB - MATHEMATICAL METHODS 3 - PAPER E - QUESTION 1

START BY FACTORIZING THE DENOMINATOR

$$f(z) = \frac{2z+1}{z^2-z-2} = \frac{2z+1}{(z+1)(z-2)}$$

$f(z)$ HAS SIMPLE POLES AT $z=-1$ & AT $z=2$

- $$\text{Res}(f;-1) = \lim_{z \to -1}\left[(z+1)f(z)\right] = \lim_{z\to -1}\left[\cancel{(z+1)}\frac{2z+1}{\cancel{(z+1)}(z-2)}\right]$$

$$= \frac{2(-1)+1}{-1-2} = \frac{-1}{-3} = \frac{1}{3}$$

- $$\text{Res}(f;2) = \lim_{z \to 2}\left[(z-2)f(z)\right] = \lim_{z\to 2}\left[\cancel{(z-2)}\frac{2z+1}{\cancel{(z-2)}(z+1)}\right]$$

$$= \frac{2\times 2+1}{2+1} = \frac{5}{3}$$

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START BY A STANDARD SUBSTITUTION

$u = -\ln x$ ← IN ORDER TO REVERSE THE ZERO LIMIT TO $+\infty$

$\frac{du}{dx} = -\frac{1}{x}$

$dx = -x\,du$

$dx = -e^{-u}\,du$ ← SINCE $-u = \ln x$, $x = e^{-u}$

AND THE LIMITS TRANSFORM

$x=0 \mapsto u=\infty$

$x=1 \mapsto u=0$

TRANSFORMING THE INTEGRAL

$$\int_0^1 (x\ln x)^n\,dx = \int_\infty^0 \left[e^{-u}(-u)\right]^n\left(-e^{-u}\,du\right) = \int_0^\infty e^{-nu}(-u)^n e^{-u}\,du$$

$$= (-1)^n\int_0^\infty e^{-nu}e^{-u}u^n\,du = (-1)^n\int_0^\infty e^{-(n+1)u}u^n\,du$$

ANOTHER SIMPLE LINEAR SUBSTITUTION TO TURN INTO A GAMMA FUNCTION

$t = (n+1)u \iff u = \frac{t}{n+1}$

$\implies du = \frac{1}{n+1}\,dt$

LIMITS UNCHANGED

$$\dots = (-1)^n\int_0^\infty e^{-t}\left(\frac{t}{n+1}\right)^n\left(\frac{1}{n+1}\,dt\right)$$

$$= (-1)^n \times \frac{1}{(n+1)^{n+1}}\int_0^\infty e^{-t}t^n\,dt$$

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USING THE DEFINITION OF THE GAMMA FUNCTION

$$\dots = \frac{(-1)^n}{(n+1)^{n+1}} \times \Gamma(n+1)$$

$$= \frac{(-1)^n}{(n+1)^{n+1}} \times n!$$

$$\therefore \int_0^1 (x\ln x)^n \, dx = \frac{(-1)^n \, n!}{(n+1)^{n+1}}$$

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LET $g(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}}$

DIFFERENTIATING g WITH RESPECT TO x

$$\frac{\partial g}{\partial x} = -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}} \times (-2t) = t(1-2xt+t^2)^{-\frac{3}{2}}$$

$$= t\left[(1-2xt+t^2)^{-\frac{1}{2}}\right]^3 = t\left[g(x,t)\right]^3$$

NEXT DIFFERENTIATE g WITH RESPECT TO t

$$\frac{\partial g}{\partial t} = -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}} \times (-2x+2t) = (x-t)(1-2xt+t^2)^{-\frac{3}{2}}$$

$$= (x-t)\left[(1-2xt+t^2)^{-\frac{1}{2}}\right]^3 = (x-t)\left[g(x,t)\right]^3$$

ADDING AND THE RESULT FOLLOWS

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} = t\left[g(x,t)\right]^3 + (x-t)\left[g(x,t)\right]^3$$

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} = x\left[g(x,t)\right]^3$$

$$\therefore \quad \frac{\partial}{\partial x}\left(g(x,t)\right) + \frac{\partial}{\partial t}\left(g(x,t)\right) = x\left[g(x,t)\right]^3$$

AS REQUIRED

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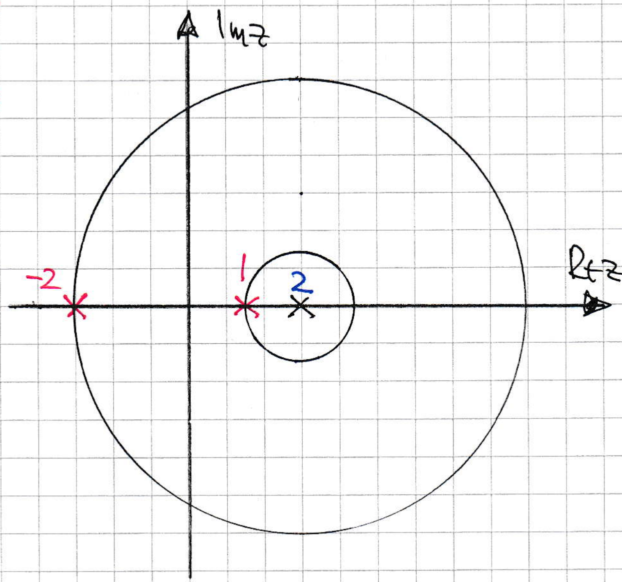
$$f(z) = \frac{1}{(z+2)(z-1)} = \frac{\frac{1}{3}}{z-1} - \frac{\frac{1}{3}}{z+2}$$



IF WE CHOOSE A CENTRE AT $z=2$, THEN

- $\frac{1}{z-1}$ CAN BE EXPANDED FOR

$$|z-2|<1 \quad \text{OR} \quad |z-2|>1$$

- $\frac{1}{z+2}$ CAN BE EXPANDED FOR

$$|z-2|<4 \quad \text{OR} \quad |z-2|>4$$

"BOTH" CAN BE EXPANDED FOR $1<|z-2|<4$ OR $|z-2|>4$

a) FIRSTLY ON THE ANNULUS $1<|z-2|<4$

- EXPAND $\frac{1}{z+2}$ FOR $|z-2|<4$

$$\frac{1}{z+2} = \frac{1}{(z-2)+4} = \frac{1}{4}\left[\frac{1}{1+\frac{z-2}{4}}\right]$$

$\left|\frac{z-2}{4}\right|<1$

$|z-2|<4$

$$= \frac{1}{4}\left[1 - \frac{z-2}{4} + \frac{(z-2)^2}{16} - \frac{(z-3)^3}{64} + \dots\right]$$

- EXPAND $\frac{1}{z-1}$ FOR $|z-2|>1$

$$\frac{1}{z-1} = \frac{1}{(z-2)+1} = \frac{1}{1+(z-2)} = \frac{1}{(z-2)\left[\frac{1}{z-2}+1\right]}$$

$\left|\frac{1}{z-2}\right|<1$

$|z-2|>1$

$$= \frac{1}{z-2}\left[\frac{1}{1+\frac{1}{z-2}}\right]$$

$$= \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \dots\right]$$

$$= \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \dots$$

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COMBINING RESULTS

$$f(z) = \frac{1}{3}\left[\frac{1}{z-1} - \frac{1}{z+2}\right]$$

$$f(z) = \frac{1}{3}\left[-\frac{1}{(z-2)^4} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2} - \frac{1}{4}\left[1 - \frac{1}{4}(z-2) + \frac{1}{16}(z-2)^2 - \frac{1}{64}(z-2)^3 + \dots\right]\right]$$

$$f(z) = \frac{1}{3}\left[\frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \dots\right] - \frac{1}{12}\left[1 - \frac{z-2}{4} + \frac{(z-2)^2}{16} - \frac{(z-2)^3}{64} + \dots\right]$$

$$f(z) = \frac{1}{3}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{(z-2)^r} - \frac{1}{12}\sum_{r=0}^{\infty}\frac{(z-2)^r}{4^r}(-1)^r$$

$$f(z) = \frac{1}{3}\sum_{r=1}^{\infty}(-1)^{r+1}(z-2)^{-r} - \frac{1}{12}\sum_{r=0}^{\infty}\left(\frac{z-2}{4}\right)^r(-1)^r$$

a) NOW WE NEED THE SAME EXPANSIONS, BOTH VALID OUTSIDE THE "BIG" CIRCLE, I.E $|z-2|>4$

- $$\frac{1}{z+2} = \frac{1}{(z-2)+4} = \frac{1}{z-2}\left[\frac{1}{1+\frac{4}{z-2}}\right]$$

$\left|\frac{4}{z-2}\right| < 1$

$|z-2| > 4$

$$= \frac{1}{z-2}\left[1 - \frac{4}{z-2} + \frac{16}{(z-2)^2} - \frac{64}{(z-2)^3} + \dots\right]$$

$$= \frac{1}{z-2} - \frac{4}{(z-2)^2} + \frac{16}{(z-2)^3} - \frac{64}{(z-2)^4} + \dots$$

- $$\frac{1}{z-1} = \frac{1}{(z-2)+1} = \frac{1}{z-2}\left[\frac{1}{1+\frac{1}{z-2}}\right]$$

$\left|\frac{1}{z-2}\right| < 1 \Rightarrow |z-2| > 1$

SO IT DEFINITELY ALSO WORKS FOR LARGER RADIUS, IE $|z-2|>4$

$$= \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \dots$$ (FOUND IN PART a)

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COLLECTING THE RESULTS

$$f(z) = \frac{1}{3}\left[\frac{1}{z-1} - \frac{1}{z+2}\right]$$

$$f(z) = \frac{1}{3}\left[\frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} - \left[\frac{1}{z-2} - \frac{4}{(z-2)^2} + \frac{16}{(z-2)^3} - \frac{64}{(z-2)^4} + \dots\right]\right]$$

$$f(z) = \frac{1}{3}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{(z-2)^r} - \frac{1}{3}\sum_{r=1}^{\infty}\frac{(-4)^{r+1}}{(z-2)^r}$$

$$f(z) = \frac{1}{3}\sum_{r=1}^{\infty}\frac{(-1)^{r+1} - (-4)^{r+1}}{(z-2)^r}$$

$$f(z) = \sum_{r=0}^{\infty}\frac{(-1)^r - (-4)^r}{3(z-2)^{r+1}}$$

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AS THE O.D.E IS ANALYTIC AT $x=0$ WE MAY TRY A SOLUTION OF THE FORM

$$y=\sum_{r=0}^{\infty}\left[a_r x^r\right]$$

DIFFERENTIATE WITH RESPECT TO x & SUBSTITUTE INTO THE O.D.E.

$$\frac{dy}{dx}=\sum_{r=1}^{\infty}\left(a_r r x^{r-1}\right) \quad \& \quad \frac{d^2y}{dx^2}=\sum_{r=2}^{\infty}\left(a_r r(r-1)x^{r-2}\right)$$

$$\Rightarrow \sum_{r=2}^{\infty}\left[a_r r(r-1)x^{r-2}\right]-x\sum_{r=0}^{\infty}\left[a_r x^{r-1}\right]=0$$

$$\Rightarrow \sum_{r=2}^{\infty}\left[a_r r(r-1)x^{r-2}\right]-\sum_{r=0}^{\infty}\left[a_r x^{r+1}\right]=0$$

EXTRACT THE LOWEST POWER OF x, IN THIS CASE x^0, OUT OF THE FIRST SUMMATION

$$\Rightarrow a_2\times 2\times 1\times x^0+\sum_{r=3}^{\infty}\left[a_r r(r-1)x^{r-2}\right]-\sum_{r=0}^{\infty}\left[a_r x^{r+1}\right]=0$$

$$\Rightarrow 2a_2+\sum_{r=0}^{\infty}\left[a_{r+3}(r+3)(r+2)x^{r+1}\right]-\sum_{r=0}^{\infty}\left[a_r x^{r+1}\right]=0$$

EQUATING POWERS YIELDS $a_2=0$ & a_0 & a_1 UNDETERMINED - FORMING A RECURRENCE RELATION FROM THE REST OF THE POWERS IN THE SUMMATIONS

$$\Rightarrow \left[a_{r+3}(r+3)(r+2)-a_r\right]x^{r+1}=0$$

$$\Rightarrow a_{r+3}(r+3)(r+2)=a_r$$

$$\Rightarrow \boxed{a_{r+3}=\frac{1}{(r+3)(r+2)}a_r}$$

USING THIS RELATION WE OBTAIN

- $r=0 \quad a_3=\frac{1}{3\times 2}a_0$
- $r=1 \quad a_4=\frac{1}{4\times 3}a_1$

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- $r=2 \quad a_5 = \frac{1}{5\times4} a_2 = 0$
- $r=3 \quad a_6 = \frac{1}{6\times5} a_3 = \frac{1}{(6\times3)(5\times2)} a_0$
- $r=4 \quad a_7 = \frac{1}{7\times6} a_4 = \frac{1}{(7\times4)(6\times3)} a_1$
- $r=5 \quad a_8 = \frac{1}{8\times7} a_5 = 0$
- $r=6 \quad a_9 = \frac{1}{9\times8} a_6 = \frac{1}{(9\times6\times3)(8\times5\times2)} a_0$
- $r=7 \quad a_{10} = \frac{1}{10\times9} a_7 = \frac{1}{(10\times7\times4)(9\times6\times3)} a_1$
- $r=8 \quad a_{11} = \frac{1}{11\times10} a_8 = 0 \qquad$ E.T.C.

WRITE THE SERIES SOLUTION FOR THE O.D.E.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$y = a_0 + a_1x + \frac{a_0}{3\times2}x^3 + \frac{a_1}{4\times3}x^4 + \frac{a_0}{(6\times3)(5\times2)}x^6 + \frac{a_1}{(7\times4)(6\times3)}x^7 + \dots$$

$$y = a_0\left[1 + \frac{1}{3\times2}x^3 + \frac{1}{(6\times3)(5\times2)}x^6 + \frac{1}{(9\times6\times3)(8\times5\times2)}x^9 + \dots\right]$$

$$+ a_1\left[x + \frac{1}{4\times3}x^4 + \frac{1}{(7\times4)(6\times3)}x^7 + \frac{1}{(10\times7\times4)(9\times6\times3)}x^{10} + \dots\right]$$

MANIPULATE FURTHER WITH GAMMA FUNCTIONS - BY LOOKING AT $[x^9]$ & $[x^{10}]$

$$[x^9]: \frac{x^9}{(9\times6\times3)(8\times5\times2)} = \frac{x^9}{3^3(3\times2\times1)\times3^3\left(\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\right)} = \frac{x^9}{9^3\times3!\times\left(\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\right)}$$

$$= \frac{x^9\times\Gamma\left(\frac{2}{3}\right)}{9^3\times3!\times\frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\times\Gamma\left(\frac{2}{3}\right)} = \frac{x^9\,\Gamma\left(\frac{2}{3}\right)}{9^3\times3!\times\Gamma\left(\frac{11}{3}\right)}$$

THE ABOVE NUMBERS IN "YELLOW" VARY WITH r, HERE $r=3$ IF WE START FROM $r=0$

$\therefore$ GENERAL TERM IS $\dfrac{x^{3r}\,\Gamma\left(\frac{2}{3}\right)}{9^r\times r!\times\Gamma\left(\frac{3r+2}{3}\right)}$

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$$[x^{10}]: \quad \frac{x^{10}}{(10\times7\times4)(9\times6\times3)} = \frac{x^{10}}{3^3\left(\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}\right)\times3^3(3\times2\times1)} = \frac{x^{10}}{9^3\times3!\times\left(\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}\right)}$$

$$= \frac{x^{10}\times\Gamma\left(\frac{4}{3}\right)}{9^3\times3!\times\frac{10}{3}\times\frac{7}{3}\times\frac{4}{3}\times\Gamma\left(\frac{4}{3}\right)} = \frac{x^{10}\,\Gamma\left(\frac{4}{3}\right)}{9^3\times3!\times\Gamma\left(\frac{13}{3}\right)}$$

IN THE ABOVE EXPRESSION THE NUMBERS IN "YELLOW" VARY WITH r, HERE $r=3$ IF WE START FROM $r=0$

$\therefore$ GENERAL TERM IS $\quad \dfrac{x^{3r+1}\,\Gamma\left(\frac{4}{3}\right)}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}$

HENCE THE GENERAL SOLUTION CAN BE WRITTEN AS

$$y = \sum_{r=0}^{\infty}\left[\frac{x^{3r}\,\Gamma\left(\frac{2}{3}\right)}{9^r\times r!\times\Gamma\left(\frac{3r+2}{3}\right)}a_0\right] + \sum_{r=0}^{\infty}\left[\frac{x^{3r+1}\,\Gamma\left(\frac{4}{3}\right)a_1}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}\right]$$

$$y = a_0\Gamma\left(\frac{2}{3}\right)\sum_{r=0}^{\infty}\left[\frac{x^{3r}}{9^r\times r!\times\Gamma\left(\frac{3r+2}{2}\right)}\right] + a_1\Gamma\left(\frac{4}{3}\right)\sum_{r=0}^{\infty}\left[\frac{x^{3r+1}}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}\right]$$

$$y = A\sum_{r=0}^{\infty}\left[\frac{x^{3r}}{9^r\times r!\times\Gamma\left(\frac{3r+2}{3}\right)}\right] + B\sum_{r=0}^{\infty}\left[\frac{x^{3r+1}}{9^r\times r!\times\Gamma\left(\frac{3r+4}{3}\right)}\right]$$

ALTERNATIVE

$$y = \sum_{r=0}^{\infty}\left[\frac{x^{3r}}{9^r\times3!}\left(\frac{A}{\Gamma\left(\frac{3r+2}{3}\right)} + \frac{Bx}{\Gamma\left(\frac{3r+4}{3}\right)}\right)\right]$$

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START BE SEEKING PARTIAL FRACTIONS USING THE SUM OF CUBES IDENTITY

$$\frac{12}{s^3+8} \equiv \frac{12}{s^3+2^3} \equiv \frac{12}{(s+2)\underbrace{(s^2-2s+4)}_{\text{IRREDUCIBLE}}} \equiv \frac{A}{s+2} + \frac{Bs+C}{s^2-2s+4}$$

$$\Rightarrow A(s^2-2s+4) + (s+2)(Bs+C) \equiv 12$$

$$\Rightarrow As^2 - 2As + 4A + Bs^2 + Cs + 2Bs + 2C \equiv 12$$

$$\Rightarrow (A+B)s^2 + (2B+C-2A)s + (4A+2C) \equiv 12$$

- if $s=-2$ $\quad A(4+4+4)=12$ (first line)
  $12A = 12$
  $A=1$
- $A+B=0 \Rightarrow B=-1$
- $4A+2C=12$
  $4+2C=12$
  $2C=8$
  $C=4.$

WE CAN NOW INVERT BY INSPECTION

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2} + \frac{-s+4}{s^2-2s+4}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{s-4}{s^2-2s+4}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = e^{-2t} - \mathcal{L}^{-1}\left[\frac{(s-1)-3}{(s-1)^2+3}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = e^{-2t} - \mathcal{L}^{-1}\left[\frac{(s-1)}{(s-1)^2+\sqrt{3}^2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+\sqrt{3}^2}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = e^{-2t} - \mathcal{L}^{-1}\left[\frac{(s-1)}{(s-1)^2+\sqrt{3}^2}\right] + \frac{3}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{(s-1)^2+\sqrt{3}^2}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right] = e^{-2t} - e^{t}\cos(\sqrt{3}t) + \sqrt{3}e^{t}\sin(\sqrt{3}t)$$

OR BY "R-TRANSFORMATION" ... $= e^{-2t} + e^{t}\left[\sqrt{3}\sin(\sqrt{3}t) - \cos(\sqrt{3}t)\right] = e^{-2t} + 2e^{t}\sin\left(\sqrt{3}t - \frac{\pi}{6}\right)$

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START BY A SUBSTITUTION, NOTING FIRST, THAT THE INTEGRAND IS EVEN IN A SYMMETRICAL DOMAIN.

$$\int_{-1}^{1}(1-x^2)^n\,dx = 2\int_0^1 (1-x^2)^n\,dx$$

$$= 2\int_0^1 (1-y)^n\left(\frac{1}{2\sqrt{y}}\,dy\right) = \int_0^1 (1-y)^n y^{-\frac{1}{2}}\,dy$$

$\sqrt{y} = x$

$y = x^2$

$dy = 2x\,dx$

$dx = \frac{dy}{2x}$

$dx = \frac{dy}{2\sqrt{y}}$

LIMITS ARE UNCHANGED

BY BETA & GAMMA FUNCTIONS

$$\ldots = B\left(n+1,\tfrac{1}{2}\right) = \frac{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}$$

$$= \frac{n!\,\sqrt{\pi}}{\Gamma\left(n+\frac{3}{2}\right)}$$

MANIPULATE FURTHER AS FOLLOWS

$$= \frac{n!\,\sqrt{\pi}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{n!\,\cancel{\sqrt{\pi}}}{\frac{1}{2}(2n+1)\times\frac{1}{2}(2n-1)\times\frac{1}{2}(2n-3)\cdots\left(\frac{1}{2}\times5\right)+\left(\frac{1}{2}\times3\right)+\left(\frac{1}{2}\times1\right)\cancel{\times\sqrt{\pi}}}$$

$$= \frac{n!}{\left(\frac{1}{2}\right)^{n+1}(2n+1)(2n-1)(2n-3)\cdots\times5\times3\times1}$$

$$= \frac{2^{n+1}\,n!\,(2n)(2n-2)(2n-4)\cdots\times6\times4\times2}{(2n+1)(2n)(2n-1)(2n-2)(2n-3)\cdots\times6\times5\times4\times3\times2\times1}$$

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$$= \frac{2^{n+1} n! \times 2n \times 2(n-1) \times 2(n-2) \times 2(n-3) \times \dots \times (2\times3)\times(2\times2)\times(2\times1)}{(2n+1)!}$$

$$= \frac{2^{n+1} \times n! \times 2^{n} \, n(n-1)(n-2)(n-3) \dots \times 3 \times 2 \times 1}{(2n+1)!}$$

$$= \frac{2^{2n+1} \times n! \times n!}{(2n+1)!}$$

$$= \frac{2^{2n+1} (n!)^2}{(2n+1)!}$$

AS REQUIRED

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a) THE FUNCTION $\frac{1}{x}$ IS NOT ABSOLUTELY INTEGRABLE IN $(-\infty,\infty)$, EVEN WITHOUT THE SINGULARITY AT $x=0$ — PROCEED AS FOLLOWS

$$\mathcal{F}\left[\frac{1}{x}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{x}e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\cancel{\frac{\cos kx}{x}}}_{\text{(ODD)}} - i\underbrace{\frac{\sin kx}{x}}_{\text{(EVEN)}}\,dx$$

DESPITE THE SINGULARITY WE WRITE AS

$$\boxed{\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\varepsilon}\frac{\cos kx}{x}\,dx + \int_{\varepsilon}^{\infty}\frac{\cos kx}{x}\,dx\right]}$$

$$\dots = -\frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\sin kx}{x}\,dx$$

THIS IS A QUOTABLE STANDARD RESULT AT THIS LEVEL

$$\dots = -\frac{2i}{\sqrt{2\pi}}\begin{cases}\frac{\pi}{2} & \text{IF } k>0\\ -\frac{\pi}{2} & \text{IF } k<0\end{cases}$$

$$= -\frac{2i}{\sqrt{2\pi}}\times\frac{\pi}{2}\times\operatorname{sign}k$$

$$= -\sqrt{\frac{\pi}{2}}\,i\operatorname{sign}k$$

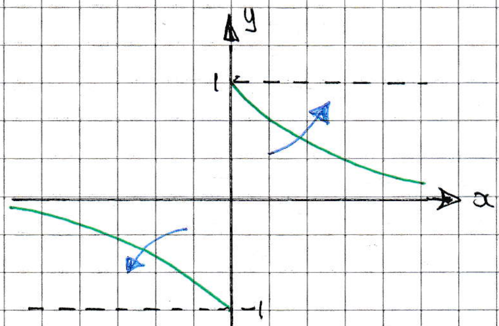
b) TRYING TO INVERT BY THE STANDARD FORMULA FAILS AS $\hat{f}(k) = -\sqrt{\frac{\pi}{2}}\,i\operatorname{sign}k$ IS NOT ABSOLUTELY INTEGRABLE IN $(-\infty,\infty)$

WE PROCEED BY THE SUGGESTED LIMITING PROCESS, SHOWN PICTORIALLY BELOW

$$\mathcal{F}^{-1}\left[-\sqrt{\frac{\pi}{2}}\,i\operatorname{sgn}(k)\right]$$

$$= -\sqrt{\frac{\pi}{2}}\,i\lim_{\varepsilon\to 0}\left[\mathcal{F}^{-1}\left[e^{-\varepsilon|k|}\operatorname{sign}k\right]\right]$$

$$= -\sqrt{\frac{\pi}{2}}\,i\lim_{\varepsilon\to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\varepsilon|k|}\operatorname{sign}k\,e^{ikx}\,dk\right]$$



$$\boxed{\hat{f}(k) = e^{-\varepsilon|k|}\operatorname{sign}k}$$

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$$= -\sqrt{\frac{\pi}{2}}\, i \times \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to 0}\left[\int_{-\infty}^{\infty} e^{-\varepsilon|k|} \operatorname{sign}k \,(\cancel{\cos kx} + i\sin kx)\,dk\right]$$

EVEN ODD EVEN ODD

$$= -\frac{1}{2} i \lim_{\varepsilon \to 0}\left[\int_{-\infty}^{\infty} e^{-\varepsilon|k|} \operatorname{sign}k \,(i\sin kx)\,dk\right]$$

$$= -\frac{1}{2} i \times 2i \times \lim_{\varepsilon \to 0}\left[\int_{0}^{\infty} e^{-\varepsilon k} \times 1 \times \sin kx \,dk\right]$$

$$= \lim_{\varepsilon \to 0}\left[\int_{0}^{\infty} e^{-\varepsilon k} \sin kx \,dk\right]$$

USING COMPLEX NUMBERS TO INTEGRATE

$$= \lim_{\varepsilon \to 0}\left[\operatorname{Im}\int_{0}^{\infty} e^{-\varepsilon k} e^{ikx}\,dk\right] = \lim_{\varepsilon \to 0}\left[\operatorname{Im}\int_{0}^{\infty} e^{(-\varepsilon + ix)k}\,dk\right]$$

$$= \lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\frac{1}{-\varepsilon + ix} e^{(-\varepsilon + ix)k}\right]_{0}^{\infty}\right]$$

$$= \lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\frac{-\varepsilon - ix}{\varepsilon^2 + x^2} e^{-\varepsilon k} \times e^{ixk}\right]_{0}^{\infty}\right]$$

$$= \lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\frac{-\varepsilon - ix}{\varepsilon^2 + x^2} e^{-\varepsilon k}(\cos kx + i\sin kx)\right]_{0}^{\infty}\right]$$

NOTE LIMITS ARE IN k

$$= \lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[0 - \frac{-\varepsilon - ix}{\varepsilon^2 + x^2} \times 1 \times (1 + 0)\right]\right]$$

$$= \lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\frac{\varepsilon + ix}{\varepsilon^2 + x^2}\right]\right]$$

$$= \lim_{\varepsilon \to 0}\left[\frac{x}{\varepsilon^2 + x^2}\right]$$

$$= \frac{x}{x^2}$$

$$= \frac{1}{x}$$ AS EXPECTED

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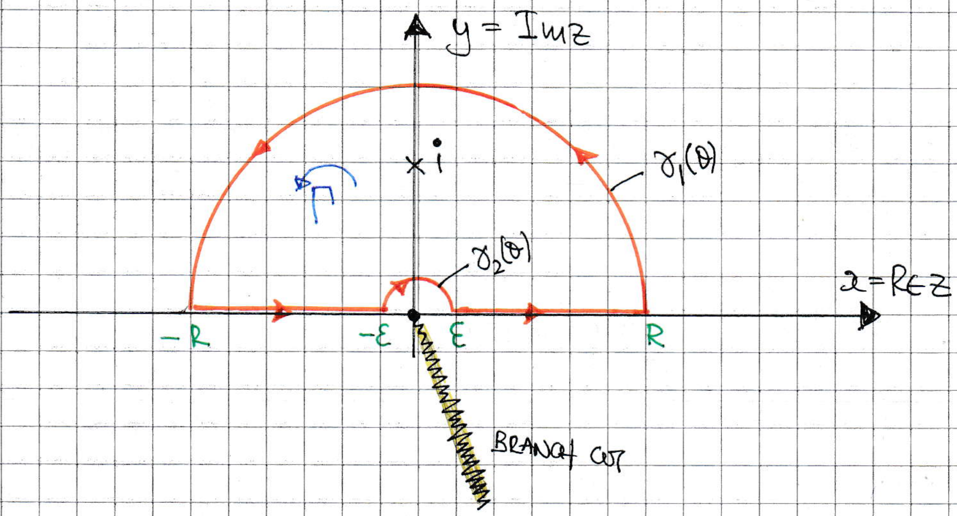
CONSIDER $f(z) = \frac{(\log z)^2}{z^2+1}$ OVER A SEMICIRCULAR CONTOUR, WHERE THE ORIGIN IS A BRANCH POINT & THE BRANCH CUT IS TAKEN ARBITRARILY IN THE 3RD OR 4TH QUADRANT.

(NOTE THAT IF THE BRANCH CUT IS TAKEN ALONG THE x-AXIS THE INTEGRATION FAILS AS THE REQUIRED INTEGRAL CANCELS OUT)



$f(z)$ HAS SIMPLE POLES AT $z = \pm i$, OF WHICH ONLY THE ONE AT $z = i$ IS INSIDE Γ — CALCULATE ITS RESIDUE

$$\lim_{z \to i}\left[(z-i)f(z)\right] = \lim_{z \to i}\left[\cancel{(z-i)}\frac{(\log z)^2}{\cancel{(z-i)}(z+i)}\right] = \frac{(\log i)^2}{2i}$$

$$= \frac{(\cancel{\log|i|} + i\arg i)^2}{2i} = \frac{\left(i\frac{\pi}{2}\right)^2}{2i} = -\frac{\pi^2}{8i}$$

BY THE RESIDUE THEOREM WE OBTAIN

$$\Longrightarrow \int_{\Gamma} f(z)\,dz = \sum(\text{RESIDUES INSIDE } \Gamma) \times 2\pi i$$

$$\Longrightarrow \left\{\int_{\varepsilon}^{R} + \int_{\gamma_1} + \int_{-R}^{-\varepsilon} + \int_{\gamma_2}\right\} f(z)\,dz = -\frac{\pi^2}{8i} \times 2\pi i = -\frac{\pi^3}{4}$$

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NOW CONSIDER THE CONTRIBUTION OF $\gamma_1(\theta)$ AS $R \to \infty$

ON $\gamma_1(\theta)$
$z = Re^{i\theta}$
$dz = iRe^{i\theta}$
$0 \leq \theta \leq \pi$

$$\left| \int_{\gamma_1} f(z)\,dz \right| \leq \int_{\gamma_1} \left| f(z)\,dz \right| = \int_{\gamma_1} \left| \frac{(\log z)^2}{1+z^2}\,dz \right|$$

$$= \int_{\gamma_1} \frac{|(\log z)^2|}{|1+z^2|}|dz| = \int_0^{\pi} \frac{\left|\left[\log(Re^{i\theta})\right]^2\right|}{|R^2e^{2i\theta}+1|}\left|iRe^{i\theta}\right| d\theta$$

$$= \int_0^{\pi} \frac{\cancel{|i|}\,|R|\,\cancel{|e^{i\theta}|}\left|\left[\log|R| + i\theta\right]^2\right|}{|R^2e^{2i\theta}+1|}\,d\theta = \int_0^{\pi} \frac{R\left|\left[\log R + i\theta\right]\right|^2}{|R^2e^{2i\theta}+1|}\,d\theta$$

NOW APPLY THE FOLLOWING INEQUALITIES

ON NUMERATOR	ON DENOMINATOR
$\lvert z \pm w\rvert \leq \lvert z\rvert + \lvert w\rvert$	$\lvert z \pm w\rvert \geq \lvert\,\lvert z\rvert - \lvert w\rvert\,\rvert$ $\frac{1}{\lvert z\pm w\rvert} \leq \frac{1}{\lvert\,\lvert z\rvert - \lvert w\rvert\,\rvert}$

$$\leq \int_0^{\pi} \frac{R\left[|\log R| + |i\theta|\right]^2}{\left||R^2e^{2i\theta}| - |1|\right|}\,d\theta = \int_0^{\pi} \frac{R\left[|\log R| + \theta\right]^2}{|R^2|\cancel{|e^{2i\theta}|} - 1}\,d\theta$$

$$= \int_0^{\pi} \frac{R\left[|\log R|^2 + 2\theta|\log R| + \theta^2\right]}{R^2-1}\,d\theta$$

$$= \frac{R}{R^2-1}\int_0^{\pi} |\log R|^2 + 2\theta|\log R| + \theta^2\,d\theta$$

$$= \frac{R}{R^2-1}\left[\theta|\log R|^2 + \theta^2|\log R| + \frac{1}{3}\theta^3\right]_0^{\pi}$$

$$= \frac{R}{R^2-1}\left[\pi|\log R|^2 + \pi^2|\log R| + \frac{\pi^3}{3}\right] = O\left[\frac{|\log R|^2}{R}\right] \to 0$$

AS $R \to \infty$

APPLY A SIMILAR LIMITING PROCESS FOR THE CONTRIBUTION OF $\gamma_2(\theta)$ AS $\varepsilon \to 0$

ON $\gamma_2(\theta)$
$z = \varepsilon e^{i\theta}$
$dz = i\varepsilon e^{i\theta}$
$0 \leqslant \theta \leqslant \pi$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leqslant \int_{\gamma_2} \left|f(z)\,dz\right| = \int_{\gamma_2}\left|\frac{(\log z)^2}{z^2+1}\,dz\right|$$

$$= \int_{\gamma_2} \frac{|(\log z)^2|}{|z^2+1|}\,|dz| = \int_{\pi}^{0} \frac{|(\log(\varepsilon e^{i\theta}))^2|}{|\varepsilon^2 e^{2i\theta}+1|}\,|i\varepsilon e^{i\theta}|\,d\theta$$

$$= \int_{\pi}^{0} \frac{\cancel{|i|}\,|\varepsilon|\,\cancel{|e^{i\theta}|}\,|[\log|\varepsilon| + i\theta]^2|}{|\varepsilon^2 e^{2i\theta}+1|}\,d\theta = -\int_{0}^{\pi} \frac{\varepsilon\,|(\log\varepsilon + i\theta)^2|}{|\varepsilon^2 e^{2i\theta}+1|}\,d\theta$$

USING THE SAME INEQUALITIES FROM PREVIOUS LIMITING PROCESS OF γ_1

$$\leqslant -\int_{0}^{\pi} \frac{\varepsilon\,[|\log\varepsilon| + |i\theta|]^2}{|\varepsilon^2 e^{2i\theta}| - |1|}\,d\theta = -\int_{0}^{\pi} \frac{\varepsilon\,[|\log\varepsilon| + \theta]^2}{|\varepsilon^2|\cancel{|e^{2i\theta}|} - 1}\,d\theta$$

$$= \frac{\varepsilon}{1-\varepsilon^2}\int_{0}^{\pi} |\log\varepsilon|^2 + 2\theta|\log\varepsilon| + \theta^2\,d\theta$$

$$= \frac{\varepsilon}{1-\varepsilon^2}\Big[\theta|\log\varepsilon|^2 + \theta^2|\log\varepsilon| + \tfrac{1}{3}\theta^3\Big]_{0}^{\pi}$$

$$= \frac{\varepsilon}{1-\varepsilon^2}\Big[\pi|\log\varepsilon|^2 + \pi^2|\log\varepsilon| + \frac{\pi^3}{3}\Big] \to 0 \text{ AS } \varepsilon \to 0$$

NOTE THAT $\varepsilon \to 0$ FASTER THAN $\log\varepsilon \to -\infty$
OR $|\log\varepsilon|^2 \to +\infty$

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SUMMARIZING THE RESULTS AS $R \to \infty$ & $\varepsilon \to 0$

$$\int_{-\infty}^{0} f(z)\,dz + \int_{0}^{\infty} f(z)\,dz = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{(\log x + i\pi)^2}{x^2+1}\,dx + \int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{(\log x + i\pi)^2 + (\log x)^2}{x^2+1}\,dx = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{(\log x)^2 + 2i\pi\log x - \pi^2 + (\log x)^2}{x^2+1}\,dx = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{2(\log x)^2 - \pi^2 + 2i\pi\log x}{x^2+1}\,dx = -\frac{\pi^3}{4}$$

$$= 2\int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx - \pi^2\int_0^\infty \frac{1}{x^2+1}\,dx + 2\pi i\int_0^\infty \frac{\log x}{x^2+1}\,dx = -\frac{\pi^3}{4}$$

ON THE POSITIVE x

$z = x = xe^{0i}$

$\log z = \log(xe^{0i})$

$\log z = \log|x| + i \times 0$

$\log z = \log|x|$

$\log z = \log x$

$0 \le x < \infty$

ON NEGATIVE x

$z = -x = xe^{i\pi}$

$\log z = \log(xe^{i\pi})$

$\log z = \log|x| + i\pi$

$\log z = \log x + i\pi$

$0 \le x < \infty$

EQUATE REAL & IMAGINARY PARTS

$$= 2\int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx - \pi^2\left(\frac{\pi}{2}\right) + 2\pi i\,\cancel{\int_0^\infty \frac{\log x}{x^2+1}\,dx}^{\text{ZERO}} = -\frac{\pi^3}{4} + 0i$$

$$= 2\int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx - \frac{\pi^3}{2} = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx - \frac{\pi^3}{4} = -\frac{\pi^3}{8}$$

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx = \frac{\pi^3}{8}$$

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START BY MANIPULATING LEGENDRE'S DUPLICATION FORMULA

$$\Gamma\left(m+\tfrac{1}{2}\right) = \frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\,\Gamma(m)}$$

$$\Gamma\left(m+\tfrac{1}{2}\right) = \frac{(2m-1)!\,\sqrt{\pi}}{2^{2m-1}(m-1)!} = \frac{2m\times(m-1)!\,\sqrt{\pi}}{2^{2m-1}\times 2\times m\,(m-1)!} = \frac{(2m)!\,\sqrt{\pi}}{2^{2m}\times m!}$$

NOW USING THE GIVEN RESULT

$$\Rightarrow \int_{-1}^{1}(1-t^2)^{n-\frac{1}{2}}e^{ixt}\,dt = \sum_{m=0}^{\infty}\left[\frac{(-1)^m x^{2m}\,\Gamma(m+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{(2m)!\,\Gamma(m+n+1)}\right]$$

$$\Rightarrow \int_{-1}^{1}\underbrace{(1-t^2)^{n-\frac{1}{2}}}_{\text{EVEN}}(\underbrace{\cos xt}_{\text{EVEN}}+i\underbrace{\sin xt}_{\text{ODD}})\,dt = \sum_{m=0}^{\infty}\left[\frac{(-1)^m x^{2m}\,\Gamma(n+\frac{1}{2})}{\cancel{(2m)!}\,\underbrace{(m+n)!}_{\Gamma(m+n+1)}}\times\underbrace{\frac{\cancel{(2m)!}\,\sqrt{\pi}}{2^{2m}\times m!}}_{\Gamma(m+\frac{1}{2})}\right]$$

TIDYING UP BOTH SIDES

$$\Rightarrow 2\int_{0}^{1}(1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \sum_{m=0}^{\infty}\left[\frac{(-1)^m\,\Gamma(n+\frac{1}{2})\sqrt{\pi}}{(m+n)!\,m!}\left(\frac{x}{2}\right)^{2m}\right]$$

$$\Rightarrow 2\int_{0}^{1}(1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(m+n)!\,m!}\left(\frac{x}{2}\right)^{2m}\right]$$

NOW THE SUMMATION IS ALMOST A BESSEL — MANIPULATE FURTHER

$$\Rightarrow 2\int_{0}^{1}(1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(m+n)!\,m!}\times\left(\frac{x}{2}\right)^{2m}\times\left(\frac{x}{2}\right)^{n}\times\left(\frac{x}{2}\right)^{-n}\right]$$

$$\Rightarrow 2\int_{0}^{1}(1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \left(\frac{x}{2}\right)^{-n}\Gamma(n+\tfrac{1}{2})\sqrt{\pi}\,\underbrace{\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(m+n)!\,m!}\left(\frac{x}{2}\right)^{2m+n}\right]}_{J_n(x)}$$

IYGB – MATHEMATICAL METHODS 3 – PAPER E – QUESTION 10

$$\Rightarrow 2\int_0^1 (1-t^2)^{n-\frac{1}{2}} \cos xt \, dt = \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\left(\frac{x}{2}\right)^{-n} J_n(x)$$

$$\Rightarrow J_n(x) = \frac{2}{\Gamma(n+\frac{1}{2})\sqrt{\pi}\left(\frac{x}{2}\right)^{-n}} \int_0^1 (1-t^2)^{n-\frac{1}{2}} \cos xt \, dt$$

$$\Rightarrow J_n(x) = \frac{2}{\Gamma(n+\frac{1}{2})\sqrt{\pi}\left(\frac{2}{x}\right)^{n}} \int_0^1 (1-t^2)^{n-\frac{1}{2}} \cos xt \, dt$$

$$\Rightarrow J_n(x) = \frac{2x^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}\, 2^n} \int_0^1 (1-t^2)^{n-\frac{1}{2}} \cos xt \, dt$$

$$\Rightarrow J_n(x) = \frac{x^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}\, 2^{n-1}} \int_0^1 (1-t^2)^{n-\frac{1}{2}} \cos xt \, dt$$

AS REQUIRED