UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH3401**
ASSESSMENT PATTERN	:	**MATH3401A**
MODULE NAME	:	**Mathematical Methods 5**
DATE	:	**20-May-14**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2013/14-MATH3401A-001-EXAM-27

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Use the method of variation of parameters to show that the solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = f(x), \quad y(0) = \frac{dy}{dx}(0) = 0,$$

is

$$y(x) = \int_0^x f(u)\left[e^{2(x-u)} - e^{(x-u)}\right] du.$$

(b) Determine a function $f(t)$ of the complex variable t and contours C_1 and C_2 in the t-plane so that

$$y_i(x) = \int_{C_i} e^{xt} f(t)\, dt, \quad i = 1, 2,$$

are independent solutions of the differential equation

$$x\frac{d^2y}{dx^2} + (6 - x)\frac{dy}{dx} - 5y = 0.$$

Show that the only solution finite at both $x = 0$ and as $x \to \infty$ is $y = 0$.

2. Consider the pair of first order differential equations

$$\frac{dx}{dt} = y - x, \qquad \frac{dy}{dt} = f(x) - y,$$

for the functions $x(t)$ and $y(t)$.

(a) Use Bendixson's negative criterion to show that there are no periodic solutions to these equations.

(b) Show that critical points are possible at $x = y = a$, where $f(a) = a$.

(c) If $f'(a) = s^2$ for real positive $s \neq 1$, show that any critical points are either saddle points or stable nodes. What is the condition on s for a critical point to be a saddle point?

(d) What is the nature of the critical point if $f'(a) < 0$?

(e) For the case $f(x) = 4x/(1 + x^2)$, make a careful sketch of the solution curves *near the origin* of the $x - y$ plane. In which direction are they traversed as t increases? Where are the other critical points and of what type are they?

3. Consider the differential equation

$$\ddot{x} + x + \varepsilon x^2 = 0,$$

for $x(t)$, $0 < \varepsilon << 1$, where a dot denotes differentiation with respect to t. Look for a periodic solution using the expansions

$$x(t) = A\cos(\theta) + \varepsilon x_1(\theta) + \cdots, \quad \theta = (1 + \varepsilon n_1 + \cdots)t,$$

where A is a positive constant.

(a) Show that

$$2\pi A n_1 = A^2 \int_{-\pi}^{\pi} \cos^3\theta \, d\theta = 0,$$

and that a suitable solution for $x_1(\theta)$ is

$$x_1(\theta) = A^2\left(\frac{\cos(2\theta)}{6} - \frac{1}{2}\right).$$

Explain why the period of the solution is independent of the value of A to $O(\epsilon)$.

(b) Extend these results to show that the period of the periodic solution to the equation

$$\frac{d^2x}{dt^2} + x + \varepsilon x^2 + \varepsilon^2 b x^3 = 0,$$

is independent of A to $O(\epsilon^2)$ if $b = 10/9$.

4. Consider the differential equation

$$\ddot{x} + x = \varepsilon f(x, \dot{x})$$

for $x(t)$, $0 < \varepsilon << 1$ where a dot denotes differentiation with respect to t.

(a) If $T = \varepsilon t$ is a slow time, show using the method of multiple scales that

$$x(t) \sim A(T)\sin(t + \Phi(T)),$$

where

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\chi f(A\sin\chi, A\cos\chi)\,\mathrm{d}\chi,$$

$$A\frac{\mathrm{d}\Phi}{\mathrm{d}T} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\chi f(A\sin\chi, A\cos\chi)\,\mathrm{d}\chi.$$

(b) For the case $f(x, \dot{x}) = \dot{x}(a^2 - x^2)$ with $a \neq 0$ and $x(0) = R > 0$, $\dot{x}(0) = o(\varepsilon)$, show that

$$x(t) \sim \frac{2aR\cos(t)}{\sqrt{R^2 + (4a^2 - R^2)\mathrm{e}^{-a^2\varepsilon t}}}.$$

Draw a graph of $x(t)$ against t for the case $R < 2a$.

5. (a) Show that, as $x \to \infty$,

(i)
$$\int_0^\infty \frac{e^{-xt}}{1+t^2}dt \sim \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{x^{2n}},$$

(ii)
$$\int_{-1}^{1} \frac{e^{-x\cosh t}}{\cosh t}\,dt \sim e^{-x}\sqrt{\frac{2\pi}{x}},$$

(iii)
$$\int_0^{\pi/4} \frac{\cos(xt^2)}{1+\tan t}\,dt \sim \sqrt{\frac{\pi}{8x}}.$$

(b) Consider the integral
$$I(x) = \int_a^b e^{x\phi(t)}f(t)\,dt.$$
If $\phi(t)$ attains its maximum on $[a, b]$ at the interior point $t = c$, so that
$$\phi'(c) = 0, \quad \phi''(c) < 0,$$
with a prime indicating differentiation with respect to t, and additionally $f(c) = 0$, $f''(c) \neq 0$, then show that
$$I(x) \sim f''(c)e^{x\phi(c)}\sqrt{\frac{\pi}{2(x|\phi''(c)|)^3}}, \qquad x \to \infty.$$