UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3401**

ASSESSMENT PATTERN : **MATH3401A**

MODULE NAME : **Mathematical Methods 5**

DATE : **06-May-11**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2010/11-MATH3401A-001-EXAM-26

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Show by direct substitution that $y(x) = \mathrm{e}^x$ is a solution to the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (1+x)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Hence, by looking for a solution of the form $y(x) = u(x)\mathrm{e}^x$, show that the solution to

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (1+x)\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2$$

is

$$y(x) = A\mathrm{e}^x + B(1+x) - x^2 - 2x - 2.$$

(b) By searching for a function $f(t)$ of the complex variable t and contours C_1 and C_2 in the complex t-plane so that

$$\int_{C_i} \mathrm{e}^{xt} f(t)\,\mathrm{d}t, \quad i = 1, 2,$$

are non trivial solutions of the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (1+x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

show that the general solution has the form

$$y(x) = A\int_0^\infty \frac{\mathrm{e}^{-xt}}{(t+1)^3}\mathrm{d}t + Bx^2\mathrm{e}^x.$$

Note: The left hand sides of the equations in parts (a) and (b) are different.

2. (a) Consider the pair of nonlinear differential equations

$$\frac{dx}{dt} = 4x - x^2 - xy, \qquad \frac{dy}{dt} = -2y + xy, \quad x \geq 0,\ y \geq 0.$$

Show that this system has three critical points consisting of two saddle points at $x = y = 0$ and at $x = 4$, $y = 0$ and a stable spiral point at $x = y = 2$. *Sketch* the phase portrait of the system.

(b) Show that the pair of equations

$$\frac{dx}{dt} = x(x^2 + y^2 - 2x - 3) - y, \qquad \frac{dy}{dt} = y(x^2 + y^2 - 2x - 3) + x,$$

can be written, using polar coordinates, as

$$\frac{dr}{dt} = r(r^2 - 2r\cos\theta - 3), \qquad \frac{d\theta}{dt} = 1,$$

and use the Poincaré-Bendixson theorem to show that the system has a limit cycle within the annulus $1 < r < 3$.

3. Consider differential equations of the general form

$$\frac{d^2x}{dt^2} + \varepsilon f(x)\frac{dx}{dt} + x = 0,$$

for $x(t)$, $\varepsilon > 0$.

(a) If $\varepsilon << 1$, then look for periodic solutions with amplitude a using the expansions

$$x(t) = a\cos(\theta) + \varepsilon x_1(\theta) + \cdots, \quad \theta = (1 + \varepsilon n_1 + \cdots)t,$$

and show that

$$n_1 = 0, \quad \int_{-\pi}^{\pi} \sin^2\theta f(a\cos\theta)\, d\theta = 0.$$

(b) For the case $f(x) = x^2 - \alpha^2$, show that such periodic solutions are possible with $a = 2\alpha$.

(c) Again with $f(x) = x^2 - \alpha^2$, but now with $\varepsilon >> 1$, use the Leinhard transformation

$$y(t) = \frac{dx}{dt} + \varepsilon F(x), \quad F(x) = \frac{1}{3}x^3 - \alpha^2 x,$$

to write the equation in the form

$$\frac{dx}{dt} = y - \varepsilon F(x), \quad \frac{dy}{dt} = -x.$$

By identifying a suitable closed trajectory in the Leinhard (x-y) plane show that periodic solutions are possible with period $\varepsilon\alpha^2(3 - 2\ln(2))$.

4. Consider the differential equation

$$\frac{d^2x}{dt^2} + x = \varepsilon f\left(x, \frac{dx}{dt}\right),$$

for $x(t)$, with $\varepsilon << 1$.

(a) If $T = \varepsilon t$ is a slow time, show using the method of multiple scales that

$$x(t) \sim A(T)\sin(t + \Phi(T)),$$

where

$$\frac{dA}{dT} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\chi f(A\sin\chi, A\cos\chi)\,d\chi,$$

$$A\frac{d\Phi}{dT} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\chi f(A\sin\chi, A\cos\chi)\,d\chi.$$

(b) If $f(x, \dot{x}) = x - \dot{x}$, $x(0) = 1$, $\dot{x}(0) = 0$, with a dot indicating differentiation with respect to t, show

$$x(t) \sim e^{-\varepsilon t/2}\cos(t - \varepsilon t/2).$$

(c) If $f(x, \dot{x}) = -\dot{x}(\dot{x} - \alpha)(\dot{x} - \beta)$, with $\alpha\beta < 0$, show that a stable limit cycle exists with

$$x(t) = 2\sqrt{\frac{|\alpha\beta|}{3}}\sin(t).$$

You may quote the result $\int_{-\pi}^{\pi} \cos^4\theta\,d\theta = 3\pi/4$.

5. (a) (i) State without proof a form of Watson's Lemma.

(ii) Values of the modified Bessel function $K_0(x)$ are given by the integral expression

$$K_0(x) = \int_1^{\infty} \frac{e^{-xt}}{\sqrt{t^2 - 1}} dt.$$

Show that for large x,

$$K_0(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}\left[1 - \frac{1}{8x}\right].$$

(iii) Show that, for large x,

$$\int_0^{\pi/4} e^{-x\sin^4 t}\sqrt{\sin t}\,dt \sim \frac{(-5/8)!}{4x^{3/8}}.$$

(b) Show that as $\nu \to \infty$ but with $x = O(1)$,

$$F(x, \nu) = \int_0^{\pi} \cos(x\cos\theta)\sin^{2\nu}\theta\,d\theta \sim \sqrt{\frac{2\pi}{\nu}}.$$