UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3401**

MODULE NAME : **Methods Of Mathematical Physics I**

DATE : **03-May-07**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH3401A-001-EXAM-27

All questions may be attempted but only marks obtained on the best **four** *solutions wi... count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let y_1 and y_2 be any two solutions of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

(i) Define the Wronskian.

(ii) What condition must hold on the Wronskian for $y_1(x)$ and $y_2(x)$ to be linearly independent solutions?

(iii) Show that the Wronskian $W(y_1, y_2)$ for the above differential equation satisfies

$$W(y_1, y_2) = Ce^{-\int P(x)dx},$$

where C is a constant of integration.

(b) Use the method of variation of parameters to show that

$$-\frac{e^{-2x}}{2}[\sin x + \cos x]$$

is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin x.$$

(c) Determine a function $f(t)$ of the complex variable t and contours C_1 and C_2 in the complex t-plane so that

$$\int_{C_i} e^{xt} f(t)dt \qquad (i = 1, 2)$$

are non-trivial solutions of the differential equation

$$x\frac{d^2y}{dx^2} + (5x+1)\frac{dy}{dx} + (6x+3)y = 0.$$

Hence find one of the solutions in closed form.

HINT: If $f(z)$ has a pole of order m at z_0, the residue at z_0 is equal to

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z).$$

2. Given that in the phase plane the local form of a differential equation near a singular point at $(x, y) = (0, 0)$ is

$$\frac{dy}{dx} = \frac{Cx + Dy}{Ax + By},$$

name and classify four types of singular points. Where applicable, state the conditions for which the singular points are stable or unstable.

What is the equivalent first order system for this differential equation?

How are the directions of the arrows along a phase plane trajectory determined?

Consider the equation of an unharmonic oscillator with a damping term:

$$\ddot{x} - \lambda x^2 + x = 0, \quad \lambda > 0,$$

where a dot denotes differentiation with respect to t and λ is a positive constant. Determine the nature of the singular points in the phase plane and show that the phase trajectories are given by

$$\frac{y^2}{2} = \frac{\lambda x^3}{3} - \frac{x^2}{2} + c$$

where $y = \dot{x}$ and c is a constant. Sketch the phase trajectories.

If at $t = 0$, $x = 0$ and $y = U$, determine that the motion is periodic if

$$U^2 < \frac{1}{3\lambda^2}.$$

Explain, without calculation, how the period of the motion can be determined.

3. Van der Pol's equation is

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$$

where a dot denotes differentiation with respect to time. Show that this can be written as

$$n^2\frac{d^2x}{d\theta^2} - \epsilon n(1 - x^2)\frac{dx}{d\theta} + x(\theta) = 0$$

where $x(\theta)$ is 2π-periodic with frequency $\frac{n}{2\pi}$.

By seeking solutions of the form

$$x(\theta) = x_0(\theta) + \epsilon x_1(\theta) + \epsilon^2 x_2(\theta) + \ldots,$$

$$n = n_0 + \epsilon n_1 + \epsilon n^2 + \ldots,$$

with $0 < \epsilon \ll 1$, show that

$$x = 2\cos(\theta) + \epsilon\left(A_1\cos(\theta) + \frac{3}{4}\sin(\theta) - \frac{1}{4}\sin(3\theta)\right) + \ldots$$

and

$$n = 1 + \epsilon^2 n_2 + \ldots$$

if $\frac{dx_i}{d\theta}(\theta = 0) = 0$ and A_1 and n_2 are unknown constants.

Derive the differential equation which holds at $O(\epsilon^2)$.

HINT: $4\sin^3\theta = 3\sin\theta - \sin 3\theta$.

4. Show that the equation

$$\ddot{x} + \epsilon f(x, \dot{x}) + x = 0,$$

where $\epsilon > 0$ is a constant and the dot denotes differentiation with respect to t, possesses a solution of the form $x = A(t)\sin[t + \phi(t)]$ if

$$\dot{A} = -\epsilon f\left(A\sin\chi, A\cos\chi\right)\cos\chi,$$
$$\dot{\phi} = \epsilon A^{-1} f\left(A\sin\chi, A\cos\chi\right)\sin\chi,$$

where $\chi = \phi + t$.

If $0 < \epsilon \ll 1$, describe a method for finding approximate solutions of these equations.

For the case of $f(x, \dot{x}) = \left(\frac{8}{3}\dot{x}^2 - 2\right)\dot{x}$ and $A(0) = A_0 > 1$, $\phi(0) = \phi_0$, show that the amplitude of the periodic solution is 1.

Show also that the general solution $x(t)$ is given approximately by

$$x = \left(1 - \left[1 - \frac{1}{A_0^2}\right]e^{-2\epsilon t}\right)^{-\frac{1}{2}} \sin\left[t + \phi_0\right].$$

Deduce the limit-cycle solution for $x(t)$.

[You may assume that $\int_0^{2\pi}\cos^2\theta d\theta = \pi$ and $\int_0^{2\pi}\cos^4\theta d\theta = \frac{3\pi}{4}$.]

5. State *without proof* a form of Watson's Lemma.

Throughout the interval $a \le t \le b$ the function $f(t)$ is continuous and the function $\phi(t)$ is decreasing from $t = a$ to $t = b$ and $\phi'(a) \neq 0$. Show that, as $x \longrightarrow +\infty$,

$$\int_a^b e^{x\phi(t)} f(t)dt \sim -\frac{e^{x\phi(a)} f(a)}{\phi'(a)x}.$$

How is this result modified if $\phi(t)$ is increasing from $t = a$ to $t = b$ and $\phi'(b) \neq 0$?

Use the results above to verify that as $x \longrightarrow +\infty$

(a) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-x\sin t}\tan t\, dt \sim -\frac{\sqrt{2}}{x}\exp\left(\frac{x}{\sqrt{2}}\right)$,

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} t^3\sin^x(t)\, dt \sim \frac{\pi^3}{64x}\exp\left(x\ln\left(\frac{1}{\sqrt{2}}\right)\right)$,

(c) $\int_{-2}^{1} e^{xt^2(t^2-1)}\, dt \sim \frac{1}{28x}e^{12x}$.