All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Where relevant, throughout the examination, you may use without proof general results from the theory of Stürm-Liouville eigenvalue problems, provided that the results used are stated clearly.

(1) The temperature distribution $u(x,t)$ in a rod of unit length and unit thermal diffusivity evolves according to the heat equation

$$u_t = u_{xx}, \qquad (t > 0, \quad 0 < x < 1).$$

Both ends of the rod radiate heat to space at the same rate, giving boundary conditions

$$\begin{aligned} u_x(0,t) &= \alpha u(0,t), \\ u_x(1,t) &= -\alpha u(1,t), \end{aligned}$$

where α is a positive constant satisfying $\alpha < \pi/2$. The initial temperature in the rod has profile $u(x,0) = T_0(x)$.

(a) Show that the temperature in the interior of the rod for $t \geq 0$ can be represented by a generalised Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} C_k \Big(p_k \cos p_k x + \alpha \sin p_k x \Big) \, e^{-p_k^2 t},$$

where the $\{p_k\}$ are a sequence of constants given by the roots of the equation

$$\tan p = \frac{2\alpha p}{p^2 - \alpha^2}.$$

(b) Illustrate graphically that there are infinitely many such roots $\{p_k\}$.

(c) Write down an expression from which the real constants $\{C_k\}$ could be evaluated. (It is not necessary to attempt to evaluate any integrals appearing in this expression.)

(2) (a) Define an *ordinary point* and a *regular singular point* of a general linear second-order ordinary differential equation.

(b) Find the regular singular points of the equation

$$(*) \qquad 2x(1+x)y'' + (1+3x)y' - 3y = 0.$$

(c) Using a Frobenius series of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+c} \qquad (a_0 = 1),$$

or otherwise, show that one solution of the equation $(*)$ is given by

$$y_1(x) = x^{1/2}\left(1 + \frac{2}{3}x - \frac{1}{5}x^2 + \frac{4}{35}x^3 - \frac{5}{63}x^4 \ldots\right),$$

and write down a general expression for a_k (the coefficient of the kth term in the series).

(d) Find the other solution $y_2(x)$. Explain briefly why the power series is truncated.

(3) Consider the eigenvalue problem

$$y'' + y' + \lambda y = 0, \quad y(0) = y(1) = 0.$$

(a) By finding the general solution of the equation and using the boundary conditions, find the eigenvalues $\{\lambda_k\}$, and show that the corresponding eigenfunctions $\{y_k(x)\}$ are given by

$$y_k(x) = e^{-x/2} \sin k\pi x.$$

(b) Write the equation in Sturm-Liouville form, and verify explicitly that the orthogonality condition on the eigenfunctions

$$\int_0^1 e^x y_j(x) y_k(x) \, dx = 0, \quad (j \neq k).$$

holds.

(c) Express the function $f(x) = e^{\alpha x}$ defined on the interval $0 \leq x \leq 1$ in a generalised Fourier series in the $\{y_k(x)\}$.

(4) The vertical displacement $u(r,t)$ of axisymmetric waves on a circular membrane of unit radius evolves according to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad (t > 0, \quad 0 \leq r < 1).$$

The edge of the membrane is held by a frame so that $u(1,t) = 0$.

(a) Use the method of separation of variables to show that the displacement for $t > 0$ is given by

$$u(r,t) = \mathrm{Re}\left(\sum_{k=1}^{\infty} A_k \exp\left(\mathrm{i}\omega_k t\right) J_0(j_{0k} r)\right),$$

where the constants $\{j_{0k}\}$ are the zeroes of the Bessel function of the first kind $J_0(\cdot)$, the constants $\{A_k\}$ are arbitrary complex constants, and the angular frequencies $\{\omega_k\}$ are to be found.

(b) If the initial velocity of the membrane $u_t(r,0)$ is zero, and its initial displacement $u(r,0)$ is given by $f(r)$, write down integral expressions involving $f(r)$ from which the constants $\{A_k\}$ can be determined.

You may state without proof that the general solution of Bessel's equation with zero index

$$xy'' + y' + xy = 0, \quad \text{is given by} \quad y(x) = AJ_0(x) + BY_0(x),$$

where A and B are arbitrary constants, and $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind, with zero index, respectively.

You may also use without proof the result of the following definite integral

$$\int_0^1 r\left(J_0(j_{0k} r)\right)^2 \, \mathrm{d}r = \frac{\left(J_1(j_{0k})\right)^2}{2}.$$

(5) A real function $f(x)$ and its Fourier transform $\hat{f}(k)$ are related through

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk.$$

(a) Find the Fourier transforms of

$$F(x) = \begin{cases} e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}, \qquad \text{and} \qquad h(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy,$$

where $\alpha > 0$ is a real constant and $f(x)$ and $g(x)$ are functions in $L^2(\mathbb{R})$.

(b) Use Fourier transforms to show that the solution of the heat equation problem

$$u_t = u_{xx}, \qquad (t > 0, \quad -\infty < x < \infty).$$

subject to the initial condition $u(x,0) = f(x)$ is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{4t}\right) dy.$$

You may quote the result

$$\int_{-\infty}^{\infty} \cos kx \, \exp\left(-\frac{x^2}{a^2}\right) dx = a\sqrt{\pi} \exp\left(-\frac{k^2 a^2}{4}\right),$$

but must prove all other results that you use.

(6) A function $f(t)$ defined on $[0,\infty)$ has a Laplace transform $\mathcal{L}[f](s) = \bar{f}(s)$ defined by

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st}\, dt.$$

(a) Given that $g(t)$ is another function defined on $[0,\infty)$, find

(i) $\mathcal{L}[e^{-\alpha t}]$ (ii) $\mathcal{L}[\dot{f}(t)]$ (iii) $\mathcal{L}\left[\int_0^t f(u) g(t-u)\, du\right]$

where α is a real constant and $\dot{f}(t)$ denotes the derivative of $f(t)$. In the case of (i) write down a restriction on s that must hold for the transform to exist.

(b) A particle with coordinates $(x(t), y(t))$, starting from the origin, evolves according to

$$\begin{aligned} \dot{x}(t) + 3x(t) + 4y(t) &= F(t) \\ \dot{y}(t) + y(t) + 2x(t) &= 0 \end{aligned}$$

where $F(t)$ is a prescribed forcing function. Using Laplace transforms, or otherwise, show that

$$y(t) = \frac{1}{3} \int_0^t F(t-u)(e^{-5u} - e^{u})\, du.$$

END OF PAPER