All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

(1) Consider Laguerre's differential equation

$$xy'' + (1-x)y' + \lambda y = 0, \qquad \lambda \text{ constant.}$$

(a) By seeking a Frobenius series solution about $x = 0$,

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+c},$$

find a recurrence relation satisfied by the first (non-logarithmic) solution, and then show that for non-integer λ the solution can be written (see Q3 for the definition of the Gamma function)

$$y_1(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)(k!)^2} x^k.$$

(b) Carefully consider what happens when $\lambda = n \geq 0$ (integer). Explain why the above formula fails, and deduce that correct solution is truncated at $k = n$. The resulting polynomial is written $y_1(x) = L_n(x)$, where $L_n(x)$ is the *Laguerre polynomial* of degree n. Show that under the usual ($a_0 = 1$) normalisation $L_n(x)$ is given by

$$L_n(x) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{(-1)^k}{k!} x^k, \qquad \text{where } \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!k!}$$

is the usual binomial coefficient.

(c) Use the Leibniz rule for differentiation to verify that another formula for $L_n(x)$ is

$$L_n(x) = \frac{e^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^n e^{-x} \right).$$

(2) The steady temperature distribution $u(r,\theta)$ in a circular wedge-shaped plate satisfies Laplace's equation

$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

in the region $0 \le r \le a$, $0 \le \theta \le \beta$. The boundaries of the plate are held at fixed temperatures given by $u(r,0) = u(r,\beta) = 0$ and $u(a,\theta) = h(\theta)$. Show that

$$u(r,\theta) = \sum_{k=1}^{\infty} A_k r^{k\pi/\beta} \sin\left(\frac{k\pi\theta}{\beta}\right),$$

$$\text{where } A_k = \alpha_k \int_0^\beta h(\theta)\sin\left(\frac{k\pi\theta}{\beta}\right)\,d\theta.$$

and the $\{\alpha_k\}$ are a sequence of real constants to be found.

(3) The Laplace transform of a function $f(t)$ defined on $[0,\infty)$ is given by

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

and the Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

(a) Calculate the Laplace transforms of

(i) $f(t) = e^{-at}$, (ii) $f(t) = t^\alpha$,

(iii) $g(t) = f(t)e^{-bt}$, (iv) $h(t) = \begin{cases} f(t-c) & t \ge c \\ 0 & t < c \end{cases}$,

where α and $c > 0$ are real and $a, b \in \mathbb{C}$. Comment on the region of validity of your solution to (i) in the complex s-plane, and on any values of α for which your solution to (ii) is undefined.

(b) Using your solutions to part (a), or otherwise, find the Laplace transform of $f(t) = t^\alpha \cosh(\beta t)\sin(\gamma t)$, using the Gamma function where necessary. Take α, β, γ to be real, and do not attempt to simplify any complex fractions. Comment on the region of validity of $\bar{f}(s)$ in the complex s-plane.

(c) Show that in the case $\alpha = 1$ the transform simplifies to

$$\bar{f}(s) = \left(\frac{\gamma(s-\beta)}{((s-\beta)^2+\gamma^2)^2} + \frac{\gamma(s+\beta)}{((s+\beta)^2+\gamma^2)^2}\right).$$

(4) Consider the differential equation given by

$$x^2(x+1)y'' + x(6x+3)y' + (6x+1)y = 0.$$

By seeking a Frobenius-type power series solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+c} \qquad (a_0 \neq 0),$$

(a) Show that the coefficients $\{a_k\}$ must satisfy a recurrence relation of the form

$$a_k = -\left(\frac{k+c+2}{k+c+1}\right) a_{k-1}$$

where the constant c takes a value to be determined.

(b) Use the recurrence relation to find a first solution $y_1(x)$ to the equation.

(c) Show that a second solution of the equation is given by

$$y_2(x) = y_1(x)\log x + \left(1 - 2x + 3x^2 - 4x^3 + \ldots\right)$$

and find the general term in the series.

(5) A real function $f(x)$ and its Fourier transform $\hat{f}(k)$ are related through

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\,\mathrm{d}x \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\,\mathrm{d}k.$$

(a) Find the Fourier transform of the convolution function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\,\mathrm{d}y.$$

(b) Using your result to (a), or otherwise, show that the solution to the upper half-plane boundary value problem

$$\begin{aligned} \nabla^2 u &= 0 && \text{in } -\infty < x < \infty,\ y > 0, \\ u(x,0) &= f(x) && \\ u(x,y) &\to 0 && \text{as } x^2 + y^2 \to \infty, \end{aligned}$$

is

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-t)}{t^2+y^2}\,\mathrm{d}t.$$

(c) Find $u(x,y)$ in the specific case where

$$f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}.$$

(6) (a) Axisymmetric solutions $u(r,\theta)$ of Laplace's equation in spherical geometry satisfy

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0.$$

Using the method of separation of variables, show that the general solution of Laplace's equation that is regular at the poles ($\theta = 0, \pi$) can be written

$$u(r,\theta) = \sum_{k=0}^{\infty}\left(A_k r^k + \frac{B_k}{r^{k+1}}\right)\mathrm{P}_k(\cos\theta).$$

[You may state without proof that the only solutions of the equation $(1-z^2)w'' - 2zw' + \nu(\nu+1)w = 0$ which are regular at $z = \pm 1$ occur for $\nu = k$ (k integer) and are the Legendre polynomials $\mathrm{P}_k(z)$.]

(b) Find the steady temperature distribution everywhere within a perfectly conducting metal sphere of unit radius when a temperature distribution

$$u(1,\theta) = T_0\cos\theta\,\sin^2\theta$$

is applied to its surface.

[Hint: Rodrigues' formula

$$\mathrm{P}_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k,$$

can be used to calculate the first few Legendre polynomials.]