UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH2401**

ASSESSMENT PATTERN : **MATH2401A**

MODULE NAME : **Mathematical Methods 3**

DATE : **05-May-10**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2009/10-MATH2401A-001-EXAM-170

TURN OVER

All questions may be answered, but only marks obtained on the best four *questions will count. The use of an electronic calculator is* not *permitted in this examination.*

1. (a) If the function $f(x)$, defined on the interval $-\pi < x < \pi$, can be written

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

show that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx,$$

where you should clearly state, *but need not derive*, any standard integrals that you use.

(b) Let

$$S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx),$$

with a_0, a_n and b_n defined as in part (a). By considering

$$\int_{-\pi}^{\pi} [f(x) - S_N(x)]^2\, dx,$$

show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx \geq 2a_0^2 + \sum_{n=1}^{N} \left(a_n^2 + b_n^2\right),$$

and deduce $a_n \to 0$, $b_n \to 0$ as $n \to \infty$.

(c) You are now given that $f(x)$ is such that $S_N(x) \to f(x)$ as $N \to \infty$. Deduce and write down Parseval's Theorem.

(d) By considering the function $f(x) = x$, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. (a) Find and classify as local maxima, minima or saddle points, the critical points of

$$f(x,y) = 3x^2 + 6xy + 2y^3 + 12x - 24y.$$

(b) Heron's formula for the area, A, of a triangle with sides a, b and c is

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad s = (a+b+c)/2.$$

Use the method of Lagrange multipliers to show that the triangle of fixed perimeter, $L = a+b+c$, with maximum area is the equilateral triangle $a = b = c = L/3$.

3. (a) Write down the Euler-Lagrange equations satisfied by the extremal curve $y(x)$ of the integral

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\,\mathrm{d}x,$$

where a dash indicates differentiation with respect to x and $y(x_1) = y_1$, $y(x_2) = y_2$. Show that, if $\partial F/\partial x = 0$, then these equations have the first integral

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}.$$

(b) Show that the extremal curve for the integral

$$\int_1^2 \left[x(y')^2 - yy'\right]\,\mathrm{d}x,$$

subject to the constraints

$$y(1) = y(2) = 0, \quad \int_1^2 y\,\mathrm{d}x = 1,$$

satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy') = \text{constant},$$

and so find this extremal curve.

4. (a) By considering the variation of the functions $x^2 + y^2 + u^2$ and $x + 2y + 3u$ on characteristics, show that the solution to the equation

$$(3y - 2u)u_x + (u - 3x)u_y = 2x - y, \quad u(x, x) = 0,$$

satisfies

$$9(x^2 + y^2 + u^2) = 2(x + 2y + 3u)^2.$$

(b) Show that the combination $x - ct$ is constant on the characteristics of the equation

$$u_t + cu_x + \lambda u = 0, \quad c, \lambda > 0,$$

satisfied by $u(x, t)$. By making a suitable change of variable, find the solution to the equation satisfying the initial condition $u(x, 0) = F(x)$.

(c) Identify the characteristic traces of the equation

$$yu_x - xu_y = c(x, y), \quad y \geq 0, \quad -1 \leq x \leq 1,$$

satisfied by $u(x, y)$ where $c(x, y)$ is an arbitrary function. Show that there is no solution to this equation with the additional condition $u(x, 0) = f(x)$ if $f(x)$ is arbitrary. Is it possible that a solution exists for a particular function $f(x)$? How could such a function $f(x)$ be generated?

5. (a) Derive D'Alembert's solution,

$$z(x, t) = \frac{1}{2}\left[F(x + ct) + F(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\, ds, \qquad (1)$$

of the one-dimensional wave equation,

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad (-\infty < x < \infty, \quad t \geq 0),$$

with c a constant, when the initial conditions are

$$z(x, 0) = F(x), \quad \frac{\partial z}{\partial t}(x, 0) = G(x), \quad (-\infty < x < \infty).$$

(b) Verify by direct calculation that (1) satisfies the correct initial conditions.

(c) If $F(x) = 0$ for $-\infty < x < \infty$ and

$$G(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0, \end{cases}$$

find $z(x, t)$ and sketch this solution for $t > 0$.

6. Consider the partial differential equation for $\theta(x,t)$, $0 \le x \le \pi$ and $t \ge 0$,

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}, \qquad \frac{\partial\theta}{\partial x}(0,t) = 0, \quad \theta(\pi,t) = 0. \tag{2}$$

(a) With a careful explanation of each step of your argument use the method of separation of variables to show that the solution has the form

$$\theta(x,t) = \sum_{n=0}^{\infty} A_n \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 t} \cos\left[\left(n+\tfrac{1}{2}\right)x\right].$$

(b) If $\theta(x,0) = 1$, show that

$$A_n = \frac{2}{\pi}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}.$$

(c) By integrating (2) with respect to x, show that

$$\frac{\partial}{\partial t}\int_0^\pi \theta(x,t)\,\mathrm{d}x = \frac{\partial\theta}{\partial x}(\pi,t) = -\frac{2}{\pi}\sum_{n=0}^{\infty} \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 t}.$$

Comment on the convergence of this sum.

(d) Solve this equation for $I(t) = \int_0^\pi \theta(x,t)\,\mathrm{d}x$ and hence show that

$$\sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2} = \frac{\pi^2}{2}.$$