

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. B.Sc.(Econ)M.Sci.

Mathematics M241: Mathematical Methods 3

COURSE CODE : MATHM241

UNIT VALUE : 0.50

DATE : 20-MAY-05

TIME : 14.30

TIME ALLOWED : 2 Hours



All questions may be attempted but only marks obtained on the best **four** solutions will count.

The use of an electronic calculator is **not** permitted in this examination.

1. Find the general solutions of the following partial differential equations

(a)

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \sin(x + y),$$

(b)

$$\frac{\partial^2 z}{\partial x^2} + 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = 0,$$

(c)

$$2x^2 z \frac{\partial z}{\partial x} + \cos^2 y \frac{\partial z}{\partial y} + xz^2 = 0.$$

In the case of (c) you may leave the solution in implicit form.

2. The functional

$$I = \int_{x_1}^{x_2} F(y, y') dx,$$

with the values of $y(x_1)$ and $y(x_2)$ prescribed, is minimised by $y(x)$. Assuming that Euler's equation is satisfied, deduce that

$$F - y' \frac{\partial F}{\partial y'} = \text{constant}, \quad (\text{Beltrami's Identity}).$$

A boat in a semi-infinite ocean ($0 < y < \infty$) is constrained to travel at speed λy . If the boat follows a path $y(x)$ between points $(0, 1)$ and $(2, 1)$, write down a functional expression for the travel time.

Show that the path which minimises the travel time satisfies the relation

$$(y'^2 + 1)y^2 = k^2, \quad (k \text{ is an unknown constant}).$$

Hence determine the minimising path and the fastest time. How does the fastest time compare to the time taken if the boat sails in a straight line between the two points?

3. Derive D'Alembert's solution

$$z(x, t) = \frac{1}{2} [F(x + ct) + F(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} G(\zeta) d\zeta$$

of the one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} \quad (-\infty < x < \infty, \quad t \geq 0)$$

with c a constant, when the initial conditions are

$$z(x, 0) = F(x), \quad z_t(x, 0) = G(x), \quad (-\infty < x < \infty).$$

If $G(x) = 0$ for $(-\infty < x < \infty)$ and

$$F(x) = \begin{cases} 1 - \sqrt{|x|}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

display the values of $z(x, t)$ for $t > 0$ in an (x, t) plane diagram.

Sketch graphs of the solution for $t = 0$, $t = 1/2c$ and $t = 2/c$.

4. Two stretched strings, each of length L , have densities of ρ_1 and ρ_2 , and each is under tension T_0 . When at rest, the strings lie along the x -axis $(-L \leq x \leq L)$ and are joined together at $x = 0$. The strings are fixed at $x = \pm L$.

Write down the partial differential equations satisfied by the displacement $z(x, t)$ in each string, if the strings perform small transverse oscillations in the plane $y = 0$. Define any constants in terms of T_0 , ρ_1 and ρ_2 , and state the boundary conditions satisfied at $x = 0$ and $x = \pm L$.

Show that the periods of the normal modes of vibration of the strings are $2\pi/\omega$ where ω satisfies the equation

$$c_1 \tan\left(\frac{\omega L}{c_1}\right) + c_2 \tan\left(\frac{\omega L}{c_2}\right) = 0,$$

with c_1 and c_2 being the wave speeds along the strings with densities ρ_1 and ρ_2 respectively.

5. (a) Determine the integral surface that satisfies the partial differential equation

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - 2z = 0,$$

and passes through the curve

$$y = 1, \quad z = 3x^2 - 8x - 3.$$

- (b) Using Lagrange multipliers, or otherwise, find the minimum distance from the hyperbola with equation

$$3x^2 - 8xy - 3y^2 = 1,$$

to the origin.

Is there a geometrical connection with the integral surface found in (a)?

6. The *steady* temperature distribution $\theta(x, y)$ of a square sheet of metal, which has its upper and lower surfaces insulated, satisfies Laplace's equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

- (a) Derive the general solution for θ in variables separable form.

- (b) Find the temperature everywhere in the sheet if the temperature is held constant ($\theta = 0$) on three edges ($y = 0$, $y = 1$ and $x = 0$) and the fourth edge at $x = 1$ is heated so that

$$\theta(1, y) = \sin 2\pi y \cos \pi y.$$

- (c) Using arguments involving symmetry and linearity, or otherwise, write down the steady solution if the edge at $y = 0$ is also heated so that

$$\theta(x, 0) = \sin 2\pi x \cos \pi x,$$

with the temperatures on the other three edges as in (b).