*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ and the vector product $\mathbf{a} \times \mathbf{b}$.
   (b) Give a geometrical interpretation of $|\mathbf{a} \times \mathbf{b}|$.
   (c) Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of two points $A$ and $B$, and let $\mathbf{p}$ be the position vector of another point $P$. Consider the line through the points $A$ and $B$ and show that the shortest distance between $P$ and this line is given by

$$d = \frac{|(\mathbf{p} - \mathbf{a}) \times (\mathbf{p} - \mathbf{b})|}{|\mathbf{b} - \mathbf{a}|}$$

2. (a) Sketch the graph of

$$f(x) = \frac{x}{1 + ax^2}, \qquad a \neq 0.$$

   Find and classify the possible extrema.
   (b) Show that $\text{arccosh(x)} = \ln\left(x + \sqrt{x^2 - 1}\right)$.
   (c) Derive the following approximation

$$\sqrt[3]{344} \approx 7 + \frac{1}{147}.$$

3. (a) Find all solutions, in the form $z = x + iy$, for real $x$ and $y$, to the equations

$$\text{i) } z^3 = 2 + 2i, \quad \text{ii) } \exp(z) = 1 - 2i, \quad \text{iii) } iz^2 + 2z = i - 1.$$

   (b) (i) If $\omega_n = \exp(2\pi i/n)$, show that

$$1 + \omega_n + \omega_n^2 + \omega_n^3 + \cdots + \omega_n^{n-1} = 0.$$

   (ii) Show

$$(x + \omega_3 y + \omega_3^2 z)(x + \omega_3^2 y + \omega_3 z) = x^2 + y^2 + z^2 - xy - yz - zx.$$

4. For $a \ll 1$ show that

$$\int_0^1 2\arctan(1 + a\ln(x))\,dx = \frac{\pi}{2} - a - a^2 - a^3 + O(a^4).$$

5. (a) Find

$$\int_0^{\pi/2} \frac{d\theta}{7 + \sin\theta}.$$

(b) Find

$$\int_0^a \text{arcsinh}(x)dx.$$

(c) Use the transformation $u = y^{-2}$ to solve the equation

$$x\frac{dy}{dx} + 2y = x^3y^3.$$

6. Solve the differential equations

(a)

$$\frac{dy}{dx} + y\tan x = \frac{x}{\cos(x)}$$

(b)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 3xe^{-x}$$

(c)

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \frac{2}{x^3}.$$



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