All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ and the vector product $\mathbf{a} \wedge \mathbf{b}$ of the two vectors $\mathbf{a}$ and $\mathbf{b}$.

   (b) Show that the volume of the parallelepiped with three concurrent (i.e. sharing a common corner) edges, given by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, is $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$ where $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c}$.

   (c) Prove that $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

2. (a) Using an Argand diagram, give a geometrical interpretation of the inequality $|z - 5i| \geq |z + i|$.

   (b) Solve the equation $z^6 = 1$ and interpret it geometrically.

   (c) By considering a geometric series (or otherwise) show that if a and θ are real numbers with $|a| < 1$ then,

$$a \sin\theta + a^2 \sin 2\theta + a^3 \sin 3\theta + \cdots = \frac{a \sin\theta}{1 - 2a\cos\theta + a^2}.$$

3. (a) Write down Leibnitz theorem expressing the nth derivative of the product $u(x)v(x)$ in terms of derivatives of $u(x)$ and $v(x)$.

   (b) Consider the function

$$y(x) = \frac{\exp(ax)}{1 - x^2}.$$

   Find the stationary points of $y(x)$ and sketch the graph of the function for the two cases $a < 0$ and $a > 0$ on a single set of axes.

   (c) Show that

$$\sinh(x/2) = \varepsilon\sqrt{\frac{1}{2}(\cosh(x) - 1)}$$

   and determine the value of ε.

4. Find the integrals

a) $\int_0^\infty x \exp(-x) \mathrm{d}x$, b) $\int_{-\infty}^1 \frac{\exp(x)}{1+2\exp(x)} \, \mathrm{d}x$, c) $\int_0^1 x \tan^{-1}(x) \, \mathrm{d}x$,

d) $\int \frac{1}{x^4+2x^2+1} \mathrm{d}x$, e) $\int \frac{\mathrm{d}x}{x^3+x^2+x}$.

5. (a) Find the solution of

$$y' = \frac{x-y}{x+y}.$$

(b) Solve the differential equation

$$y'' + y' - 6y = x + \exp(2x).$$

6. Determine the series solutions for the following differential equation

$$(x^2-1)y'' - 8xy' + 20y = 6x^2$$

about the regular point $x_0 = 0$.