MATH1401 2012

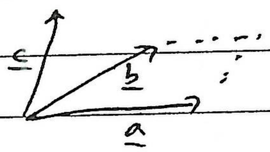
1) (a) $\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta$

$|\cdot|$ denotes the norm, θ is the angle between $\underline{a}$ and $\underline{b}$

$\underline{a} \wedge \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta\, \hat{\underline{c}}$

where $\hat{\underline{c}}$ is a unit vector normal to $\underline{a}$ & $\underline{b}$ and choosen such that $\underline{a}, \underline{b}, \hat{\underline{c}}$ form a right-handed set

(b)



The area spanned by $\underline{a}$ and $\underline{b}$ is $|\underline{a} \wedge \underline{b}|$

The vector normal to this area is $\dfrac{\underline{a} \wedge \underline{b}}{|\underline{a} \wedge \underline{b}|}$ and thus

the height of the parallelepiped is $\underline{c} \cdot \dfrac{\underline{a} \wedge \underline{b}}{|\underline{a} \wedge \underline{b}|}$.

Therefore the volume is

$$V = \left| \, |\underline{a} \wedge \underline{b}| \, \underline{c} \cdot \frac{\underline{a} \wedge \underline{b}}{|\underline{a} \wedge \underline{b}|} \right| = |(\underline{a} \wedge \underline{b}) \cdot \underline{c}|$$

(c) $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\quad \underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\underline{b} \wedge \underline{c} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \wedge \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_2 c_3 - b_3 c_2 \\ b_3 c_1 - b_1 c_3 \\ b_1 c_2 - b_2 c_1 \end{pmatrix}$$

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \wedge \begin{pmatrix} b_2 c_3 - b_3 c_2 \\ b_3 c_1 - b_1 c_3 \\ b_1 c_2 - b_2 c_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 (b_1 c_2 - b_2 c_1) - a_3 (b_3 c_1 - b_1 c_3) \\ a_3 (b_2 c_3 - b_3 c_2) - a_1 (b_1 c_2 - b_2 c_1) \\ a_1 (b_3 c_1 - b_1 c_3) - a_2 (b_2 c_3 - b_3 c_2) \end{pmatrix}$$

$$= \left(\begin{array}{l} a_2c_2b_1 + a_3c_3b_1 - a_2b_2c_1 - a_3b_3c_1 + a_1c_1b_1 - a_1c_1b_1 \\ a_3c_3b_2 + a_1c_1b_2 - a_3b_3c_2 - a_1b_1c_2 + a_2b_2c_2 - a_2b_2c_2 \\ a_1c_1b_3 + a_2c_2b_3 - a_1b_1c_3 - a_2b_2c_3 + a_3b_3c_3 - a_3b_3c_3 \end{array} \right.$$

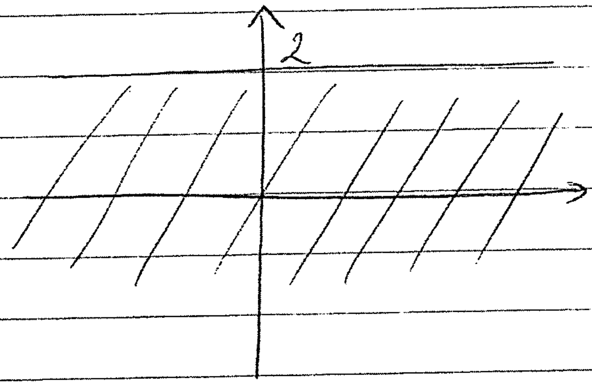
$$= (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$$

New model solution 2) (a) & (b)

beginning the same as in original 2) (a) (ii)

$|x+i(y-5)| \geq |x+i(y+1)|$

$\Rightarrow 2 \geq y$ $\mathbb{C}$



2) (b) $z^6 = 1$

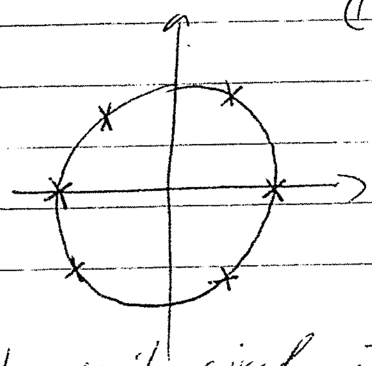
Write $z = e^{i\theta}$ and $1 = e^{2\pi i k}$ $k \in \mathbb{Z}$

$$z^6 = e^{6i\theta} = e^{2\pi i k} \Rightarrow \theta = \frac{2\pi k}{6} = \frac{\pi}{3}k$$

$k=0$	$\theta = 0$	$z = 1$
$k=1$	$\theta = \pi/3$	$z = \cos\pi/3 + i\sin\pi/3$
$k=2$	$\theta = 2\pi/3$	$z = \cos 2\pi/3 + i\sin 2\pi/3$
$k=3$	$\theta = \pi$	$z = -1$
$k=4$	$\theta = 4\pi/3$	$z = \cos 4\pi/3 + i\sin 4\pi/3$
$k=5$	$\theta = 5\pi/3$	$z = \cos 5\pi/3 + i\sin 5\pi/3$

Higher values of k do not give additional roots

$\mathbb{C}$



Roots of unity part the unit circle into equal segments.

$z = a(\cos\theta + i\sin\theta)$ $\qquad z^n = a^n(\cos n\theta + i\sin n\theta)$

$$\text{Im}\left[1 + z + z^2 + z^3 + \ldots\right] = \text{Im}\left[\frac{1}{1-z}\right]$$

$$a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \ldots$$

$$= \text{Im}\left[\frac{1}{1 - a\cos\theta - ai\sin\theta}\right]$$

$$= \text{Im}\left[\frac{1}{1-a\cos\theta - ai\sin\theta} \cdot \frac{1-a\cos\theta + ai\sin\theta}{1-a\cos\theta + ai\sin\theta}\right]$$

$$= \text{Im}\left[\frac{1-a\cos\theta + ai\sin\theta}{(1-a\cos\theta)^2 + a^2\sin^2\theta}\right]$$

$$= \frac{a\sin\theta}{1 - 2a\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta} = \frac{a\sin\theta}{1-2a\cos\theta + a^2}$$

3) (a) $(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$

(b) $y(x) = \dfrac{e^{ax}}{1-x^2}$

$$y'(x) = \frac{ae^{ax}(1-x^2) - e^{ax}(-2x)}{(1-x^2)^2}$$

$$y' = 0 \iff e^{ax}\left\{a(1-x^2) + 2x\right\} = 0 \qquad e^{ax} \neq 0$$

$$-x^2 + \frac{2}{a}x + 1 = 0$$

$$x^2 - \frac{2}{a}x - 1 = 0$$

$$x_{1/2} = \frac{1}{a} \pm \sqrt{\frac{1}{a^2}+1} = \frac{1}{a}\left\{1 \pm \sqrt{1+a^2}\right\}$$

$x = 1+\varepsilon$

$$y = \frac{e^{a+\varepsilon a}}{1-(1+\varepsilon)^2} = \frac{e^{a}e^{\varepsilon a}}{1-1-2\varepsilon-\varepsilon^2}$$

$$= -\frac{e^{a}e^{\varepsilon a}}{2\varepsilon+\varepsilon^2} \overset{\varepsilon \ll 1}{\simeq} -\frac{\text{const.} > 0}{2\varepsilon} \;;\; \text{const} > 0$$

$x = -1+\varepsilon$

$$y = \frac{e^{-a}e^{\varepsilon a}}{1-(\varepsilon-1)^2} = \frac{e^{-a}e^{\varepsilon a}}{1-\varepsilon^2+2\varepsilon-1} = \frac{e^{-a}e^{\varepsilon a}}{2\varepsilon-\varepsilon^2}$$

$$\overset{\varepsilon \ll 1}{\simeq} \frac{\text{const.}}{2\varepsilon} \;;\; \text{const} > 0$$

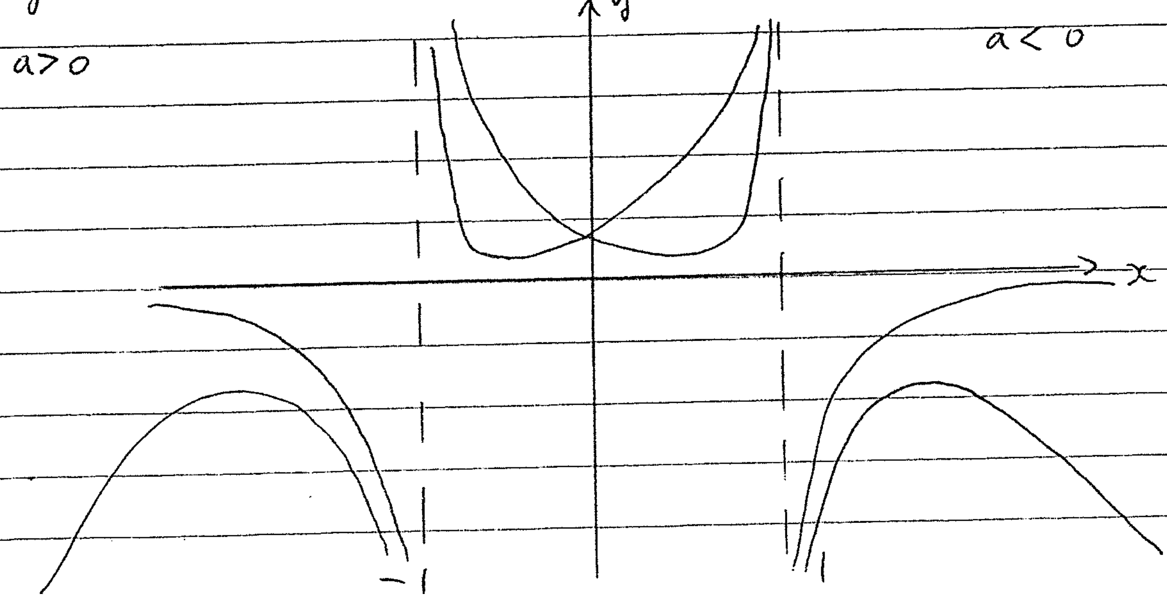
$a > 0$ $\quad y(x \text{ large}) \sim -e^{ax}$

$\qquad\quad y(-x \text{ large}) \sim -e^{-ax} \longrightarrow 0$

$a < 0$ $\quad y(x \text{ large}) \sim -e^{-ax} \longrightarrow -0$

$\qquad\quad y(-x \text{ large}) \sim -e^{ax}$

$y(x)$ cannot cross the x-axis



(c) $\cosh^2 x = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$

$\sinh^2 x = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$

$\sinh^2 x + \cosh^2 x = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \cosh 2x$

$\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow \cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x$

$2\sinh^2 x = \cosh 2x - 1$

$\sinh x = \sqrt{\frac{1}{2}(\cosh 2x - 1)}$

$\Rightarrow \sinh \frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x - 1)}$ Thus $\varepsilon = +1$

4) a) $\int_0^\infty x e^{-x}\, dx = -x e^{-x}\Big|_0^\infty - \int^\infty -e^{-x}\, dx$

$= \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1$

b) $\int_{-\infty}^{1} \frac{e^x}{1+2e^x}\, dx = \frac{1}{2}\int_{-\infty}^{1} \frac{2e^x}{1+2e^x}\, dx = \frac{1}{2}\ln\left|1+2e^x\right|\Big|_{-\infty}^{1}$

$= \frac{1}{2}\ln|1+2e|$

c) $\int_0^1 x\tan^{-1}x\, dx = \frac{1}{2}x^2\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{\frac{1}{2}x^2}{1+x^2}\, dx$

$= \frac{1}{2}x^2\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{1+x^2}\, dx$

$$= \frac{1}{2}x^2\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 1 - \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2\tan^{-1}x\Big|_0^1 - \frac{1}{2}x\Big|_0^1 + \frac{1}{2}\tan^{-1}x\Big|_0^1$$

$$= \frac{1}{2}\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{2}$$

d) $$\int \frac{dx}{x^4+2x^2+1} = \int \frac{dx}{(x^2+1)^2}$$

(partial fractions or) $x = \tan u$

$$\frac{1}{(x^2+1)^2} = \frac{1}{(\tan^2 u+1)^2} = \frac{1}{\left(1/\cos^2 u\right)^2} = \cos^4 u$$

$$\frac{dx}{du} = \frac{1}{\cos^2 u} \Rightarrow dx = \frac{du}{\cos^2 u}$$

$$\int\frac{dx}{(x^2+1)^2} = \int \frac{du/\cos^2 u}{\cos^{-4} u} = \int \cos^2 u\,du$$

$$= \frac{1}{2}\int 1+\cos 2u\,du = \frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right) + C_1$$

$$= \frac{1}{2}u + \frac{1}{2}\sin u\cos u + C_1$$

$$\sin(\tan^{-1}x) = \frac{x}{\sqrt{1+x^2}} \qquad \cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2}$$

$$\int \frac{dx}{x(\underbrace{x^2+x+1}_{\text{no real roots}})} \qquad x_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}$$

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$1 = A(x^2+x+1) + Bx^2 + Cx$$

$x=0 \qquad 1 = A$

$x^2: \qquad 0 = A + B \qquad B = -A$

$x: \qquad C + A = 0 \qquad A = -C$

$$A = 1 \; ; \; B = -1 \; ; \; C = -1$$

$$\int \frac{dx}{x^3+x^2+x} = \int \frac{1}{x} - \frac{x+1}{x^2+x+1} \, dx$$

$$= \ln x - \frac{1}{2} \int \frac{2x+1+1}{x^2+x+1} \, dx$$

$$= \ln x - \frac{1}{2} \ln(x^2+x+1) - \frac{1}{2} \int \frac{1}{x^2+x+1} \, dx$$

$$(x+\tfrac{1}{2})^2 = x^2 + x + \tfrac{1}{4}$$

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} = 2 \int \frac{dx}{(2x+1)+3}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right) + C_1$$

$$\Rightarrow \int \frac{dx}{x^3+x^2+x} = \ln x - \frac{1}{2} \ln(x^2+x+1) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1+2x}{\sqrt{3}}\right) +$$

5) (a) $$y' = \frac{x-y}{x+y} = \frac{1-y/x}{1+y/x}$$

$$z = y/x \Rightarrow y = xz \Rightarrow y' = z + xz'$$

$$\Rightarrow xz' + z = \frac{1-z}{1+z}$$

$$xz' = \frac{1-z}{1+z} - z = \frac{1-z-z(1+z)}{1+z} = \frac{1-2z-z^2}{1+z}$$

$$\int dz \frac{1+z}{1-2z-z^2} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln|1-2z-z^2| = \ln x + C_1$$

$$\ln(1-2z-z^2) = -2\ln x + \tilde{C}_1$$

$$-1+2z+z^2 = -\hat{C}_1 \frac{1}{x^2}$$

$$\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1 + \frac{\hat{C}_1}{x^2} = 0$$

$$\left(y/x\right) = -1 \pm \sqrt{1+1-\frac{\hat{C}_1}{x^2}}$$

$$y = x\left(-1 \pm \sqrt{2 - \hat{C}_1/x^2}\right)$$

$$= -x \pm \sqrt{2x^2 + \tilde{C}}$$

$\tilde{C}$ and the sign follow from initial conditions.

(b) $y'' + y' - 6y = x + e^{2x}$

$\lambda^2 + \lambda - 6 = 0 \qquad \lambda_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 6}$

$= -\frac{1}{2} \pm \frac{5}{2}$

$\lambda_1 = -3 \qquad \lambda_2 = 2$

C.F. $\quad y(x) = Ae^{-3x} + Be^{2x}$

P.I. $\quad y(x) = \alpha + \beta x + \gamma x e^{2x}$

$y'(x) = \beta + \gamma e^{2x} + \gamma x 2e^{2x}$

$y''(x) = \gamma 2e^{2x} + \gamma 2e^{2x} + \gamma x 4e^{2x}$

$\cancel{4\gamma x e^{2x}} + 4\gamma e^{2x} + \gamma e^{2x} + \cancel{\gamma x 2e^{2x}} + \beta$

$-6\alpha - 6\beta x - \cancel{6\gamma x e^{2x}} = x + e^{2x}$

$5\gamma e^{2x} + \beta - 6\alpha - 6\beta x = x + e^{2x}$

$5\gamma = 1 \Rightarrow \gamma = \frac{1}{5} \; ; \quad \beta - 6\alpha = 0 \Rightarrow \beta = 6\alpha$

$-6\beta = 1 \Rightarrow \beta = -\frac{1}{6}$

$\alpha = \frac{\beta}{6} = -\frac{1}{36}$

complete solution

$y = Ae^{-3x} + Be^{2x} - \frac{1}{36} - \frac{1}{6}x + \frac{1}{5}xe^{2x}$

6) $$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$y'(x) = \sum_{k=1}^{\infty} a_k k x^{k-1}$$

$$y''(x) = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

$$(x^2-1)y'' - 8xy' + 20y - 6x^2 = 0$$

$$\sum_{k=2}^{\infty} a_k k(k-1)x^k - \sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - 8\sum_{k=1}^{\infty} a_k k x^k$$
$$+ 20\sum_{k=0}^{\infty} a_k x^k - 6x^2 = 0$$

① ✓

② $a_2 2(1)x^0 + a_3 3(2)x + \sum_{k=4}^{\infty} a_k k(k-1)x^{k-2}$

$k-2=\tilde{k}$ $\qquad \sum_{\tilde{k}=2}^{\infty} a_{\tilde{k}+2}(\tilde{k}+2)(\tilde{k}+1)x^{\tilde{k}}$

③ $a_1(1)x + \sum_{k=2}^{\infty} a_k k x^k$

④ $a_0 x^0 + a_1 x^1 + \sum_{k=2}^{\infty} a_k x^k$

$$\Rightarrow \sum_{k=2}^{\infty} a_k k(k-1)x^k - 2a_2 - 6a_3 x - \sum_{k=2}^{\infty} a_{k+2}(k+2)(k+1)x^k$$
$$-8a_1 x - 8\sum_{k=2}^{\infty} a_k k x^k + 20a_0 + 20a_1 x + 20\sum_{k=2}^{\infty} a_k x^k$$
$$-6x^2 = 0$$

x^0 terms $\quad -2a_2 + 20a_0 = 0 \implies a_2 = 10a_0$

x^1 terms $\quad -6a_3 - 8a_1 + 20a_1 = 0$

$\iff -6a_3 + 12a_1 = 0 \implies a_3 = 2a_1$

x^2 terms $\quad \underline{a_2}\,2(1)x^2 - a_4\,4(3)x^2 - 8\underline{a_2}(2)x^2 + 20\underline{a_2 x^2} - 6x^2 = 0$

$$6a_2 - 12a_4 - 6 = 0$$

$$a_2 - 2a_4 - 1 = 0 \implies 2a_4 = a_2 - 1$$

$$\implies a_4 = \frac{1}{2}(a_2 - 1) = \frac{1}{2}(10a_0 - 1)$$

x^k terms, $k \geq 3$ $\quad a_k k(k-1) - a_{k+2}(k+1)(k+1) - 8a_k k + 20a_k = 0$

$$a_k\{k(k-1) - 8k + 20\} = a_{k+2}(k+2)(k+1)$$

$$a_{k+2} = \frac{k^2 - 9k + 20}{(k+2)(k+1)}a_k = \frac{(k-4)(k-5)}{(k+2)(k+1)}a_k$$

If $k=3$ $\quad a_5 = \frac{(-1)(-2)}{(5)(4)}a_3 = \frac{1}{5\cdot 2}2a_1 = \frac{1}{5}a_1$

$k=4 \quad a_6 = 0 \quad \& \quad k=5 \quad a_7 = 0$

Therefore all subsequent terms vanish!

$$\implies y(x) = a_0 + a_1 x + 10a_0 x^2 + 2a_1 x^3 + \frac{1}{2}(10a_0 - 1)x^4 + \frac{1}{5}a_1 x^5$$

As expected, there are two constants of integration a_0 & a_1