University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2012
Mathematics 1401
Thursday 12 January 2012 2:30 –4:30 or 4:15 – 6:15

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Carefully state the definition of the vector product $\mathbf{a} \wedge \mathbf{b}$ of the two vectors $\mathbf{a}$ and $\mathbf{b}$.

   (b) Write $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$ and show that

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) + \mathbf{c} \wedge (\mathbf{a} \wedge \mathbf{b}) + \mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{a}) = \mathbf{0}.$$

   (c) Show that a necessary condition for the equation $\mathbf{a} \wedge \mathbf{x} = \mathbf{b}$ to have a solution for the unknown vector $\mathbf{x}$ is $\mathbf{a}.\mathbf{b} = 0$. Now suppose this condition is satisfied. Write $\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{a}\wedge\mathbf{b}$ and calculate α, β, γ. Show that the set of solutions represents a line parallel to the vector $\mathbf{a}$ and write down the minimum distance of this line from the origin.

2. (a) State De Moivre's theorem and use it to show that if $z = \exp(i\theta)$ then

$$(z^n + z^{-n}) = 2\cos(n\theta) \quad \text{and} \quad (z^n - z^{-n}) = 2i\sin(n\theta),$$

   where $n \geq 0$ is an integer.

   (b) Hence show that

$$\sin^5\theta = \tfrac{1}{16}\left(\sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta)\right).$$

   (c) Hence, or otherwise, evaluate

$$\int_0^{\pi} \theta \sin^4(\theta)\cos(\theta)\, \mathrm{d}\theta.$$

PLEASE TURN OVER

3. (a) Show

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) = -\frac{a}{x\sqrt{x^2-a^2}}.$$

(b) Write down Leibniz's theorem expressing the nth derivative of the product $u(x)v(x)$ in terms of derivatives of $u(x)$ and $v(x)$. Use it to find the nth derivative of $f(x) = (x^2+1)\exp(ax)$. Hence show

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x^2+1)\cos x = (x^2-11)\cos x + 8x\sin x.$$

4. (a) Evaluate

$$\int_0^\infty \frac{x\,\mathrm{d}x}{(x+1)^2(x^2+1)}, \quad \int_0^\infty \frac{\tan^{-1}(x)}{1+x^2}\,\mathrm{d}x.$$

(b) If

$$I_n = \int_{\pi/2}^{x} \frac{\cos^{2n+1} t}{\sin t}\,\mathrm{d}t, \quad n \geq 0,$$

show that

$$2(n+1)I_{n+1} = 2(n+1)I_n + \cos^{2n+2} x, \quad n > 0.$$

Hence find I_3.

5. (a) Find the solution of

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + (x-2)y = x^4, \quad y(1) = 1.$$

(b) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 12y = \cosh 3x.$$

6. Determine the series solutions for the following differential equation

$$(x^2+1)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$$

about the regular point $x_0 = 0$.

END OF PAPER