University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2011
Mathematics 1401
Thursday 13 January 2011 2.30 –4.30 or 4.15 – 6.15

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$. Give a careful definition of their vector product $\mathbf{a} \wedge \mathbf{b}$.

(b) Show that the distance between the two skew lines $\mathbf{r}_1 = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu \mathbf{b}_2$ is

$$\left| \frac{(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2)}{|(\mathbf{b}_1 \wedge \mathbf{b}_2)|} \right|.$$

(c) Show that if the two lines

$$\frac{x - c_1}{d_1} = \frac{y - c_2}{d_2} = \frac{z - c_3}{d_3}, \qquad \frac{x - d_1}{c_1} = \frac{y - d_2}{c_2} = \frac{z - d_3}{c_3},$$

intersect, they lie in the plane

$$\mathbf{r} \cdot (\mathbf{c} \wedge \mathbf{d}) = 0,$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ and $\mathbf{c}$ and $\mathbf{d}$ are not parallel vectors.

2. (a) State De Moivre's theorem and use it to show that if $z = \exp(i\theta)$ then

$$(z^n + z^{-n}) = 2\cos(n\theta) \quad \text{and} \quad (z^n - z^{-n}) = 2i\sin(n\theta),$$

where $n \geq 0$ is an integer.

(b) Hence show that

$$\cos^4 \theta = \frac{1}{8}(\cos(4\theta) + 4\cos(2\theta) + 3).$$

(c) Hence, or otherwise, evaluate

$$\int_0^\pi \theta \cos^3(\theta) \cos(\theta)\, d\theta.$$

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3. The function defined by

$$\mathrm{gd}(t) = \arctan(\sinh t)$$

is known as the *gudermannian* of t.

(a) Show that gd is odd and sketch its graph.

(b) Given that

$$\arctan(u) = u - \frac{1}{3}u^3 + \frac{1}{5}u^5 + \cdots,$$

$$\sinh t = t + \frac{1}{6}t^3 + \frac{1}{120}t^5 + \cdots,$$

find the power series for $\mathrm{gd}(x)$ up to and including terms in x^5.

(c) Show that $\mathrm{gd}(t) = \int_0^t \operatorname{sech} u \, \mathrm{d}u$.

4. Show that for $\varepsilon \ll 1$

$$\int_0^{\pi/4} \tan(\varepsilon \tan x) \, \mathrm{d}x = \frac{1}{6}\left(\log(8)\varepsilon + (1 - \log(2))\varepsilon^3 + \cdots\right)$$

5. (a) Find b, such that

$$\int_0^\infty \frac{x^2 + bx}{(x+1)^2(x^2+1)} \, \mathrm{d}x = \frac{1+\pi}{6}.$$

Evaluate

$$\int_0^\infty \frac{\tan^{-1}(x)}{1+x^2} \, \mathrm{d}x.$$

(b) If

$$I_n = \int_{\pi/2}^{x} \frac{\cos^{2n+1} t}{\sin t} \, \mathrm{d}t, \quad n \geq 0, \quad \pi/2 < x < \pi,$$

show that

$$2(n+1)I_{n+1} = 2(n+1)I_n + \cos^{2n+2} x, \quad n \geq 0.$$

Hence find $I_3(x)$.

6. Find the solutions of

$$xy' + (x-2)y = x^5, \quad y(1) = 1,$$

and

$$x^2y'' - 2xy' + 2y = \ln x, \quad y(1) = y'(1) = 1$$

where a dash denotes differentiation with respect to x.

END OF PAPER