UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1401**

ASSESSMENT PATTERN : **MATH1401A**

MODULE NAME : **Mathematical Methods 1**

DATE : **07-May-10**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2009/10-MATH1401A-001-EXAM-217

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a}.\mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$.

   (b) Two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ have an angle γ between them. Show that the vector $\hat{\mathbf{b}} - (\hat{\mathbf{a}}.\hat{\mathbf{b}})\hat{\mathbf{a}}$ is perpendicular to $\hat{\mathbf{a}}$ and of length $|\sin\gamma|$.

   (c) In a triangle ABC the perpendiculars from B and C to the opposite sides meet at D. Prove that $\vec{\mathbf{AC}}.(\vec{\mathbf{DA}} + \vec{\mathbf{AB}}) = \vec{\mathbf{AB}}.(\vec{\mathbf{DA}} + \vec{\mathbf{AC}}) = 0$, and hence that $\vec{\mathbf{DA}}.\vec{\mathbf{BC}} = 0$, showing that the perpendiculars from the all the vertices to opposite sides meet at D.

2. (a) Define $\sinh x$, $\cosh x$ and $\tanh x$ in terms of the exponential function. Show

$$\cosh^2 x - \sinh^2 x = 1, \qquad \frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}.$$

   (b) If $y(x) = \operatorname{arcsinh}^2 x$, show that

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2.$$

   (c) If $u^n(x)$ denotes the nth derivative of $u(x)$ and $y(x)$ is as in part (b), show

$$(x^2+1)y^{n+2}(x) + (2n+1)xy^{n+1}(x) + n^2y^n(x) = 0, \qquad n > 0.$$

3. (a) Define the Taylor series of a sufficiently smooth function $f(x)$ in a neighbourhood of x_0. Show that near $x_0 = 0$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5).$$

   (b) Compute the Taylor series of $1/f(x)$ near $x_0 = 0$ up to $O(x^5)$, $f(x_0) \neq 0$. Find the Taylor series of $1/\cos x$ near $x_0 = 0$ up to $O(x^5)$.

   (c) Combining parts (a) and (b) show that

$$\cos x \times \frac{1}{\cos x} = 1 + O(x^5).$$

4. (a) Use that fact that $\text{Im}(\exp(iax)) = \sin ax$ to find

$$\int_0^\infty \exp(bx)\sin ax\, dx,$$

where a and b are real. For what values of b does the integral exist?

(b) If $z = \cos\theta + i\sin\theta$, show by expanding $(z+z^{-1})^4(z-z^{-1})^4$ that

$$\sin^4\theta\cos^4\theta = \frac{1}{2^7}(3 - 4\cos(4x) + \cos(8x)).$$

Hence, or otherwise, show

$$\int_0^\pi \sin^4\theta\cos^4\theta\, d\theta = \frac{3\pi}{128}.$$

5. Find the integrals

(a)

$$\int \frac{\sin^3 x}{\sqrt{\cos x}}\, dx.$$

(b)

$$\int \frac{dx}{\sqrt{x+1}+\sqrt{x}}.$$

(c)

$$\int \frac{1-\cos x}{\sin x}\, dx.$$

6. Solve the differential equations

(a)

$$\frac{dy}{dx} = \frac{y+2}{x-2}, \quad y(0) = 1.$$

(b)

$$(x+1)\frac{dy}{dx} + xy = \frac{e^{-x}}{x+1}.$$

(c)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + 2\sinh x.$$

 END OF PAPER