University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2010
Mathematics 1401
Wednesday 13 January 2010 2.30 –4.30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination*

1. (a) Let a plane be given by $\mathbf{r} = \mathbf{a} + \lambda_1 \mathbf{h}_1 + \lambda_2 \mathbf{h}_2$. Show that it can be written as the vector equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$. Compute $\mathbf{n}$. What is the meaning of $\mathbf{a} \cdot \hat{\mathbf{n}}$?

   (b) Show that the minimum distance between the two lines $\mathbf{r}_1 = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu \mathbf{b}_2$ is given by

$$\left| \frac{(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2)}{|\mathbf{b}_1 \wedge \mathbf{b}_2|} \right|$$

   (c) Show that the point $(0, 0, 1)$ lies in both of the planes $x + y + z = 1$ and $x - y + 3z = 3$ and find the minimum distance between the x-axis and the line of intersection of these two planes.

2. (a) Sketch the graph of $y(x) = \exp(-3x)/(1 - x)$.

   (b) Show that $\frac{d}{dx} \operatorname{arctanh} x = 1/(1 - x^2)$.

   (c) Show that the nth derivative of $y = (x^2 + 1)\exp(2x)$ is

$$\frac{d^n y}{dx^n} = 2^{n-2} \exp(2x)(4x^2 + 4nx + n^2 - n + 4).$$

3. (a) Define the Taylor series of a sufficiently smooth function $f(x)$ in a neighbourhood of x_0. Show that near $x_0 = 0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6).$$

   (b) Use the above series to derive the Maclaurin series for $\sinh(x)$, $\cosh(x)$ and $\cosh(\sinh(x))$ up to $O(x^6)$.

   (c) Show that $\cosh(x)$ has a minimum at $x = 0$.

   (d) Show that $\cosh^2(x) - \sinh^2(x) = 1 + O(x^5)$.

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4. (a) What do the following sets represent in the complex plane?

$$\text{i)}\quad |z| = |z - 3 + 2i|, \qquad \text{ii)}\quad |z - 2i| + |z + 2i| = 6.$$

(b) Find $(1+i)^{19}$ and $(-\sqrt{3}+i)^{31}$.

(c) Find all z such that $z^5 - 1 = 0$ and mark these in the complex plane.

5. (a) If $I_n = \int \sinh^n x\, dx$, show that

$$nI_n = \cosh x \sinh^{n-1} x - (n-1)I_{n-2}, \qquad n \geq 2.$$

Hence or otherwise find $\int x \cosh x \sinh^5 x\, dx$.

(b) Find

$$\int \frac{d\theta}{1 - \tan\theta}.$$

6. Solve the differential equations

(a) $\frac{dy}{dx} + y \cot x = x$, $y(\pi/2) = 1$,

(b) $\frac{dy}{dx} = 2xy/(x^2 + y^2)$,

(c) $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 3/x^4$.

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