UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1401**
ASSESSMENT PATTERN	:	**MATH1401A**
MODULE NAME	:	**Mathematical Methods 1**
DATE	:	**06-May-09**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2008/09-MATH1401A-001-EXAM-193

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions wi count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors,

   (i) Define the scalar product $\mathbf{a}\cdot$

   (ii) Give a careful definition of the vector product $\mathbf{a}\wedge\mathbf{b}$,

   (iii) If a, b and c are the position vectors, relative to some origin, of three points A, B and C respectively, show that the area of the triangle ABC is

$$\tfrac{1}{2}|\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a}|.$$

   (b) In a triangle ABC the perpendiculars from B and C to the opposite sides meet at D. Prove that $\vec{\mathbf{AC}}.(\vec{\mathbf{DA}}+\vec{\mathbf{AB}}) = \vec{\mathbf{AB}}.(\vec{\mathbf{DA}}+\vec{\mathbf{AC}}) = 0$, and hence that $\vec{\mathbf{DA}}.\vec{\mathbf{BC}} = 0$. Deduce that the perpendiculars from the vertices to the opposite sides of a triangle are concurrent.

2. (a) Find all solutions, in the form $z = x + iy$, for real x and y, to the equations

$$\text{i) } z^3 = 2+i, \quad \text{ii) } iz^2 + 2z = i - 1, \quad \text{iii) } \exp(z) = 1 + i.$$

   (b) (i) If $\omega_n = \exp(2\pi i/n)$, show that $\omega_n^n = 1$ and that

$$1 + \omega_n + \omega_n^2 + \omega_n^3 + \cdots + \omega_n^{n-1} = 0.$$

   (ii) Show

$$(x + \omega_3 y + \omega_3^2 z)(x + \omega_3^2 y + \omega_3 z) = x^2 + y^2 + z^2 - xy - yz - zx.$$

3. (a) (i) Express $\sinh x$, $\cosh x$ and $\tanh x$ in terms of the exponential function.

   (ii) Show that two solutions of $\cosh x = a$, $a > 1$, are $x = \ln(a \pm \sqrt{a^2-1})$. Show that the sum of these two solutions is zero and comment.

   (iii) Show

$$\int \frac{dx}{\cosh x} = 2\tan^{-1}(\exp(x)) + \text{constant}.$$

   (b) Show that

$$\frac{d}{dx}\tan^{-1}\left(\frac{a+x}{1-ax}\right)$$

   is independent of the value of a and explain this observation.

4. (a) Evaluate the integrals

$$\text{i)}\ \int (3x-4)^{3/2}\,\mathrm{d}x, \quad \text{ii)}\ \int_0^{\pi/3} \frac{\sin x}{\cos^4 x}\,\mathrm{d}x, \quad \text{iii)}\ \int_0^{\pi/4} \tan^n x \sec^2 x\,\mathrm{d}x.$$

(b) Use the substitution $x - p = 1/u$ to show

$$\int \frac{\mathrm{d}x}{(x-p)\sqrt{(x-p)(x-q)}} = \frac{2}{q-p}\sqrt{\frac{x-q}{x-p}} + \text{constant}.$$

(c) If $m \neq -1$ and n is a positive integer, show

$$\int x^m (\ln x)^n \mathrm{d}x =$$

$$x^{m+1}\left(\frac{(\ln x)^n}{(m+1)} - \frac{n(\ln x)^{n-1}}{(m+1)^2} + \frac{n(n-1)(\ln x)^{n-2}}{(m+1)^3} - \cdots + \frac{(-1)^n n!}{(m+1)^{n+1}}\right) + \text{constant}.$$

5. Solve the differential equations

(a) $(1-x^2)y' = x(a-y)$,

(b) $x^2y' = x^2 + 3xy + y^2$,

(c) $xy' + 2y = \sqrt{1+3x^2}, \quad y(1) = 1.$

where a dash denotes differentiation with respect to x.

6. Solve the differential equations

(a) $y'' - 4y' + 3y = 8\exp(-x) - 2\exp(x), \qquad y(0) = 1, \quad y'(0) = 0,$

(b) $x^2y'' + xy' + y = \sin(\ln(x))$.

where a dash denotes differentiation with respect to x.