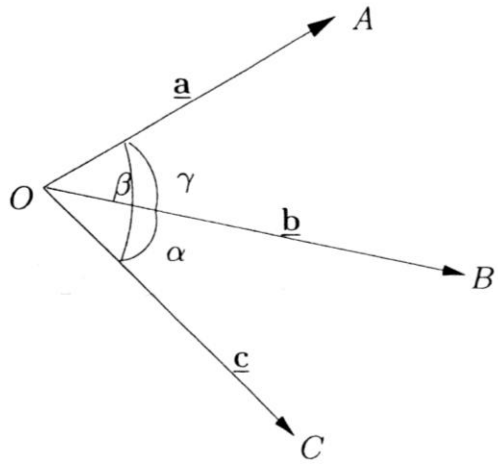
University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2009
Mathematics 1401
Friday 16 January 2009 10.30 – 12.30 or 12.15 – 2.15

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. The points A, B and C are each *unit distance* from O and the position vectors of these points relative to O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The angles between the pairs of vectors $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{a}$, $\mathbf{a}$ and $\mathbf{b}$ are α, β and γ respectively. These angles are all less than $\pi/2$.



Show that the vectors $\mathbf{u} = \mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{a}$ and $\mathbf{v} = \mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{a}$ are both normal to the vector $\mathbf{a}$ and lie in the planes OBA and OCA respectively. Show also that $|\mathbf{u}| = \sin\gamma$ and $|\mathbf{v}| = \sin\beta$. The angle between the planes OBA and OCA is ϕ. Show that

$$\cos\phi = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

Using the fact that the vectors

$$\mathbf{w} = \mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{b} \quad \text{and} \quad \mathbf{z} = \mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{b}$$

have magnitudes $\sin\alpha$ and $\sin\gamma$ respectively and are both normal to $\mathbf{b}$, show that

$$\frac{\sin\phi}{\sin\alpha} = \frac{\sin\psi}{\sin\beta},$$

where ψ is the angle between the planes OCB and OAB. [*Hint:* Consider $(\mathbf{v}\wedge\mathbf{u})\cdot\mathbf{a}$ and $(\mathbf{z}\wedge\mathbf{w})\cdot\mathbf{b}$.]

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2. (a) Find the least value of $|z|$ if $|z-i| = |z-2|$

(b) State De Moivre's theorem and use it to

(i) express $\cos 6\theta$ as a polynomial in $\cos\theta$

(ii) and hence, making use of the subsitution $x = \cos^2\theta$, show that the roots of the equation $64x^3 - 96x^2 + 36x - 3 = 0$ are $\cos^2(\pi/18)$, $\cos^2(5\pi/18)$, $\cos^2(7\pi/18)$.

3. (a) Show that the function

$$f(x) = \frac{\exp(ax)}{1+\exp(x)},$$

has a stationary point at

$$x = \ln\left(\frac{a}{1-a}\right), \quad f(x) = a^a(1-a)^{1-a},$$

for $0 < a < 1$. Sketch the graph of the function for the three cases $a = \frac{1}{2}$, $a = 1$, $a = 2$ on a single set of axes.

(b) The function $y(x)$ satisfies the differential equation

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2, \quad \frac{dy}{dx}(0) = 1.$$

Use Leibniz's theorem to differentiate this equation n times and so find an expression for $\frac{d^my}{dx^m}(0)$ for m odd and for m even.

4. (a) Show that

$$\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\sec x = \sec x \tan x.$$

Hence show that

$$\int \sec x\, dx = \ln(\sec x + \tan x) + \text{constant}.$$

(b) For which values of α, β and γ do the following converge

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx, \quad \int_0^\infty \frac{x^\gamma \tanh x\, dx}{1+x^2} \quad ?$$

(c) Show

$$\int \frac{dx}{\cos x + a\sin x} = \frac{1}{\sqrt{1+a^2}}\ln\left(\frac{\sqrt{1+a^2} - a + \tan(x/2)}{\sqrt{1+a^2} + a - \tan(x/2)}\right).$$

CONTINUED

5. (a) Find the general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x(y^2+1)$$

and the solution with $y(0) = 1$. Give a rough sketch of the solution for $0 \le x < \frac{1}{2}\sqrt{\pi}$.

(b) Find the solution to

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2, \qquad y(1) = 0.$$

6. Solve the differential equations

(a) $y'' - 7y' + 6y = \sinh x + 6x$,

(b) $x^2y'' - 3xy' + 4y = x^3$, $y(1) = 2$, $y'(1) = 2$,

where a dash denotes differentiation with respect to x.

END OF PAPER