UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1401**

ASSESSMENT PATTERN : **MATH1401A**

MODULE NAME : **Mathematical Methods 1**

DATE : **13-May-08**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2007/08-MATH1401A-001-EXAM-210

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) If $\mathbf{a}$, $\mathbf{b}$ are vectors,
   (i) Define the scalar product $\mathbf{a}\cdot\mathbf{b}$,
   (ii) Give a careful definition of the vector product $\mathbf{a}\wedge\mathbf{b}$,
   (iii) Show that $a^2b^2 = \mathbf{a}\cdot\mathbf{b}\ \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}\cdot\mathbf{a}\wedge\mathbf{b}$ where a is the magnitude of the vector $\mathbf{a}$.

   (b) A plane contains the three points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Derive an expression for the distance of a fourth point, with position vector $\mathbf{r}$, from this plane.

   (c) A plane contains the point $(1, -2, 1)$ and the line with parametric equation $\mathbf{r} = (1+3\lambda)\mathbf{i} + (\lambda-1)\mathbf{j} - 2\lambda\mathbf{k}$. Find the minimum distance of this plane from the origin.

2. (a) State De Moivre's theorem and use it to
   (i) evaluate
   $$\left(1+\sqrt{3}i\right)^{10} - \left(1-\sqrt{3}i\right)^{10}$$
   and
   (ii) show that if $z = \exp(i\theta)$ and m an integer, then
   $$(z^m + z^{-m}) = 2\cos(m\theta) \quad \text{and} \quad (z^m - z^{-m}) = 2i\sin(m\theta).$$

   (b) Find the n solutions of $z^n = 1$ with n an odd positive integer and plot them in the Argand diagram. Show that their sum is zero and hence deduce that
   $$\sum_{m=1}^{\frac{1}{2}(n-1)} \cos\left(\frac{2m\pi}{n}\right) = -\frac{1}{2}.$$

3. (a) Sketch graphs of the functions $\sinh x$, $\cosh x$, $\sinh^{-1} x$ and $\cosh^{-1} x$ on the same set of axes.

(b) Show that the derivative of $y(x) = \sinh^{-1} x$ is $(1+x^2)^{-1/2}$. By differentiating $\ln(x+\sqrt{1+x^2})$, show $\sinh^{-1} x = \ln(x+\sqrt{1+x^2})$.

(c) State Leibnitz' theorem for nth derivative of the product $u(x)v(x)$. Use it to show

$$\frac{\mathrm{d}^n x^2 \mathrm{e}^{\alpha x}}{\mathrm{d}x^n} = \alpha^{n-2}\mathrm{e}^{\alpha x}\left(\alpha^2 x^2 + 2n\alpha x + n(n-1)\right).$$

Hence show that, if m is a positive even integer,

$$\frac{\mathrm{d}^m \left(x^2\cosh \alpha x\right)}{\mathrm{d}x^m} = \alpha^{m-2}\left(x^2 \sinh \alpha x + 2\alpha n x \cosh \alpha x + n(n-1)\sinh \alpha x\right).$$

What is the corresponding result if m is odd.

4. (a) Find

i) $\displaystyle\int x\sec^2 x \,\mathrm{d}x$, ii) $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^r x \,\mathrm{d}x$, iii) $\displaystyle\int \ln(\ln x)\frac{\ln x}{x}\,\mathrm{d}x$,

if $r > -1$.

(b) If

$$S_m = \int x^m \sin nx \,\mathrm{d}x \quad \text{and} \quad C_m = \int x^m \cos nx \,\mathrm{d}x,$$

show that

$$nS_m = -x^m \cos nx + mC_{m-1} \quad \text{and} \quad nC_m = x^m \sin nx - mS_{m-1}.$$

Hence find

$$\int_0^{\pi} x^5 \sin x \,\mathrm{d}x.$$

5. (a) Find the solution of

$$(x+a)\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = (x+a)^4.$$

(b) Interchange the dependent and independent variables to show that the solution to

$$(2x - 5y^3)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \qquad y(1) = 1,$$

is $y = x^{1/3}$.

(c) Use the substitution $z(x) = (y(x))^{-2}$ to find the general solution to the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy - y^3 \mathrm{e}^{-x^2}.$$

6. (a) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^2 + x, \qquad y(0) = \frac{\mathrm{d}y}{\mathrm{d}x}(0) = 0.$$

(b) Use the substitution $1 + x = \mathrm{e}^t$ to solve the differential equation

$$(1+x)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1+x)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 4\cos(\ln(1+x)).$$