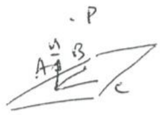
MATH 1401 (2008)

1. a) i) $\underline{a}\cdot\underline{b} = ab\cos\theta$ with $a = |\underline{a}|$ & θ the angle between $\underline{a}$ & $\underline{b}$

ii) $\underline{a}\wedge\underline{b} = ab\sin\theta\,\hat{\underline{n}}$ with $\hat{\underline{n}}$ a unit vector $\perp$ to $\underline{a}$ & $\underline{b}$ & in such a direction that $\underline{a}, \underline{b}, \underline{a}\wedge\underline{b}$ form a right-handed set.

iii) $\underline{a}\cdot\underline{b}\,\underline{a}\cdot\underline{b} + \underline{a}\wedge\underline{b}\cdot\underline{a}\wedge\underline{b} = a^2b^2\cos^2\theta + a^2b^2\sin^2\theta\,\hat{\underline{n}}\cdot\hat{\underline{n}} = a^2b^2$

b) 

Normal to plane, $\hat{n}$, is $\vec{AC}\wedge\vec{AB} = (\underline{c}-\underline{a})\wedge(\underline{b}-\underline{a})$. The distance is the projection of $\vec{AP}$ onto $\hat{n}$ ie $(\underline{r}-\underline{a})\cdot\dfrac{(\underline{c}-\underline{a})\wedge(\underline{b}-\underline{a})}{|(\underline{c}-\underline{a})\wedge(\underline{b}-\underline{a})|}$

c) If $A = (1,-2,1)$, $B = (1,-1,0)$ $(\lambda=0)$ & $C = (4,0,-2)$ $(\lambda=1)$ then with $P = (0,0,0)$, the formula gives, $\underline{c}-\underline{a} = \begin{pmatrix}3\\2\\-3\end{pmatrix}$, $\underline{b}-\underline{a} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$, $\underline{n} = \begin{pmatrix}3\\2\\-3\end{pmatrix}\wedge\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\3\\3\end{pmatrix}$, $|\underline{n}| = \sqrt{19}$ & distance is $-\frac{1}{\sqrt{19}}\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-3\end{pmatrix} = \underline{4/\sqrt{19}}$.

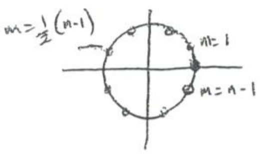
2. a) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

i) $z = (1+\sqrt{3}i)^{10}$, $w = 1+\sqrt{3}i$, required is $z - \bar{z} = 2i\,\mathrm{Im}(z) = 2i|w|^{10}\sin 10\theta$, $\theta = \mathrm{Arg}(w)$

$|w| = \sqrt{1+3} = 2$, $\theta = \tan^{-1}\sqrt{3} = \pi/3$, $\sin 10\pi/3 = -\sin\pi/3 = -\sqrt{3}/2$. Ans is $\underline{-2^{10}\sqrt{3}i}$

ii) $z = \cos\theta + i\sin\theta$, $z^m = \cos m\theta + i\sin m\theta$, $z^{-m} = \cos m\theta - i\sin m\theta$, (sin is odd, cos is even), $z^m + z^{-m} = 2\cos m\theta$, $z^m - z^{-m} = 2i\sin m\theta$.

b) If $z^n = 1$ & $z = re^{i\theta}$, then $r^ne^{in\theta} = 1e^{2\pi im} \Rightarrow r = 1, \theta = m(2\pi/n)$, $m = 0, 1, \ldots n-1$



If $w = e^{i2\pi/n}$, roots are $1, w, w^2, \ldots w^{n-1}$. If Sum is S, then $S - wS = 1 - w^n = 1 - 1 = 0$, so as $w \neq 1$, $S = 0$

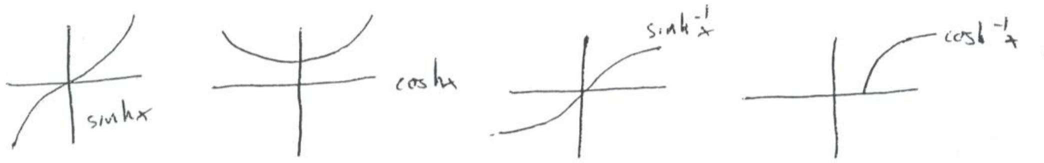
Taking real parts of sum gives

$$1 + \underset{m=1}{\cos\frac{2\pi}{n}} + \underset{m=2}{\cos\frac{4\pi}{n}} + \underset{m=3}{\cos\frac{6\pi}{n}} + \cdots + \cos\frac{(n-2)2\pi}{n} + \underset{m=n-1}{\cos\frac{(n-1)2\pi}{n}} = 0$$

but as $\cos\theta = \cos(2\pi-\theta)$ we have

$$1 + 2\sum_{m=1}^{\frac{n-1}{2}}\cos\frac{m2\pi}{n} = 0$$ & so the result.

3. a)



b) If $y = \sinh^{-1}x$, then $\sinh y = x \Rightarrow \cosh y \, dy/dx = 1$, $\Rightarrow dy/dx = 1/\cosh y = 1/\sqrt{1+x^2}$

$$\frac{d}{dx}\ln(x+\sqrt{1+x^2}) = \frac{1}{x+\sqrt{1+x^2}}\left\{1+\frac{2x\cdot\frac{1}{2}}{\sqrt{1+x^2}}\right\} = \frac{1}{x+\sqrt{1+x^2}}\cdot\frac{(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

So $\sinh^{-1}x$ & $\ln(x+\sqrt{1+x^2})$ have same derivative & so differ by a constant. At $x=0$ they are both zero & hence they are the same.

c) $(uv)^n = \sum_{r=0}^{n} {}^nC_r u^r v^{n-r}$. So $(x^2 e^{\alpha x})^n = x^2\alpha^n e^{\alpha x} + 2x.n.\alpha^{n-1}e^{\alpha x} + 2.\frac{n(n-1)}{2}\alpha^{n-2}e^{\alpha x}$

as $d^n e^{\alpha x}/dx^n = \alpha^n e^{\alpha x}$

$= \alpha^{n-2}e^{\alpha x}\left[\alpha^2 x^2 + 2n\alpha x + n(n-1)\right]$

& using $\cosh\alpha x = \frac{1}{2}(e^{\alpha x}+e^{-\alpha x})$ the required result is

$$\frac{1}{2}\left[\alpha^{m-2}e^{\alpha x}\left[\alpha^2x^2+2n\alpha x+n(n-1)\right] + (-\alpha)^{m-2}e^{-\alpha x}\left[(-\alpha)^2x^2 - 2n\alpha x + n(n-1)\right]\right]$$

& if m is even $(-\alpha)^m = \alpha^m$ & we have $\alpha^{m-2}\left[\alpha^2x^2\cosh\alpha x + 2n\alpha x\sinh\alpha x + (n)(n-1)\cosh\alpha x\right]$

if m is odd $(-\alpha)^m = -\alpha^m$ & we have $\alpha^{m-2}\left[\alpha^2x^2\sinh\alpha x + 2n\alpha x\cosh\alpha x + n(n-1)\sinh\alpha x\right]$

4. a) i) $\int x\sec^2 x\,dx = [x\tan x] - \int\tan x\,dx = x\tan x + \ln\cos x + C$

ii) $\int_0^{\pi/2}\sin^3 x\cos^m x\,dx = \int_0^{\pi/2}\sin x\cos^m x - \sin x\cos^{m+2}x\,dx = \left[-\cos^{m+1}x/_{m+1} + \cos^{m+3}x/_{m+3}\right]_0^{\pi/2}$

$= 1/_{m+1} - 1/_{m+3}$.

iii) $\int \ln(\ln x)\ln x/x\,dx = \int u\ln u\,du = u^2/_2\cdot\ln u - u^2/_4 + C = \frac{1}{2}(\ln x)^2\left[\ln(\ln x) - 1/_2\right] + C$

$u = \ln x$, $du = \frac{dx}{x}$

b) $S_m = \int x^m\sin nx\,dx = \left[-x^m\frac{\cos nx}{n}\right] + \int m x^{m-1}\frac{\cos nx}{n}\,dx \Rightarrow nS_m = -x^m\cos nx + mC_{m-1}$

$C_m = \int x^m\cos nx\,dx = \left[x^m\frac{\sin nx}{n}\right] - \int m x^{m-1}\frac{\sin nx}{n}\,dx \Rightarrow nC_m = x^m\sin nx - mS_{m-1}$

So $(n=1)$ $\int_0^\pi x^5\sin x\,dx = \left[-x^5\cos x\right]_0^\pi + 5\left\{\left[x^4\sin x\right]_0^\pi - 5S_3\right\} = \pi^5 - 25S_3$

$\int_0^\pi x^3\sin x\,dx = \left[-x^3\cos x\right]_0^\pi + 3\left\{\left[x^2\sin x\right] - 3S_1\right\} = \pi^3 - 9S_1$

& $\int_0^\pi x\sin x\,dx = \left[-x\cos x\right]_0^\pi + \int_0^\pi\cos x\,dx = \pi$

& answer is $\pi^5 - 25\pi^3 + 25.9.\pi$.

5. a) $dy/dx - \frac{3}{x+a} y = (x+a)^3$. IF is $e^{-3\ln(x+a)} = 1/(x+a)^3 \Rightarrow \frac{d}{dx}\left[y/(x+a)^3 \right] = 1$

$\Rightarrow y = \underline{(x+A)(x+a)^3}$

b) $\frac{dy}{dx} = \frac{y}{5y^3 - 2x} \Rightarrow \frac{dx}{dy} = 5y^2 - \frac{2x}{y}$, $\frac{dx}{dy} + \frac{2x}{y} = 5y^2$. IF is $y^2 \Rightarrow \frac{d}{dy}\left[xy^2 \right] = 5y^4$

$\Rightarrow xy^2 = y^5 + C$, $y(1) = 1 \Rightarrow C = 0$ & $y^3 = x$, $\underline{y = x^{1/3}}$

c) $\frac{dy}{dx} = xy - y^3 e^{-x^2}$. If $z = y^{-2}$, then $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$ & multiplying by $-2y^{-3}$ gives

$\frac{dz}{dx} = -2zx + 2e^{-x^2}$ ie $\frac{dz}{dx} + 2xz = 2e^{-x^2}$. IF is e^{x^2} & $\frac{d}{dx}\left[e^{x^2} z \right] = 2$

$z = (2x + A)e^{-x^2}$, $\underline{y = \pm e^{x^2/2} / \sqrt{2x+A}}$.

6. a) $y'' - 3y' - 4y = x^2 + x$, a.e. $m^2 - 3m - 4 = 0 = (m-4)(m+1)$, $m = 4, -1$. CF $Ae^{4x} + Be^{-x}$

PI $y = \alpha x^2 + \beta x + \gamma$ with ; on substitution, $-4\alpha = 1$, $-3\alpha - 4\beta = 1$, $2\alpha - 3\beta - 4\gamma = 0$

& $\alpha = -1/4$, $\beta = -1/16$, $\gamma = -1/4\left[+ 1/2 - 3/16 \right] = -5/4.16$

$y = CF + PI$ & $y(0) = 0 \Rightarrow A + B + \gamma = 0$, $y'(0) = 0 \Rightarrow 4A - B = \beta \Rightarrow 5A = \beta - \gamma = \frac{1}{16}\left(\frac{1}{4}\right)$

& $B = 5/4.16 - 1/5.4.16 = 24/5.4.16$

$$\underline{y = (1/5.4.16)\left[-5.16x^2 - 5.4x - 5.5 + e^{4x} + 24e^{-x} \right]}$$

b) $(1+x) = e^t \Rightarrow t = \ln(1+x)$, $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{1}{1+x} \Rightarrow (1+x)\frac{dy}{dx} = \frac{dy}{dt}$, $\frac{d^2y}{dx^2} = \frac{1}{(1+x)^2}\frac{d^2y}{dt^2} - \frac{1}{(1+x)^2}\frac{dy}{dt}$

$\Rightarrow d^2y/dt^2 + y = 4\cos t$. CF $y = A\cos t + B\sin t$, PI try

$y = t(\alpha\cos t + \beta\sin t)$ & substitution requires

$2(-\alpha\sin t + \beta\cos t) + t(-\alpha\cos t + \beta\sin t) + t(\alpha\cos t + \beta\sin t) = 4\cos t$

$\Rightarrow \alpha = 0$, $\beta = 2$

& $y = A\cos t + B\sin t + 2t\sin t$

$$\underline{y = A\cos\ln(1+x) + B\sin\ln(1+x) + 2\ln(1+x)\sin\ln(1+x)}$$