1401
JAN 08

University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2008
Mathematics 1401
Wednesday 9 January 2008 2.30 – 4.30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination*

1. (a) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors.

   (i) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$,

   (ii) Give a careful definition of the vector product $\mathbf{a} \wedge \mathbf{b}$,

   (iii) Define the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

   (b) By considering the volume of the parallelepiped formed by the three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, prove that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}]$ and use this result to show that the scalar and the vector products may be interchanged in the definition of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

   (c) Show that the scalar product of $\mathbf{r}$ with $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) - \mathbf{a} \wedge \mathbf{b} - \mathbf{a} \wedge \mathbf{c}$ is zero *for any vector* $\mathbf{r}$. Deduce that $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$.

2. (a) State De Moivre's theorem and use it to show that

$$\frac{\sin 4\theta}{\sin \theta} = 8\cos^3\theta - 4\cos\theta = 2\cos 3\theta + 2\cos\theta.$$

   Comment on these equations as $\theta \to 0$.

   (b) The points z_1, z_2, z_3 form an equilateral triangle in the Argand diagram. If $z_1 = 4 + 6i$ and $z_2 = (1 - i)z_1$, find the two possible positions of z_3.

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3. (a) Sketch the curves

$$\text{i)}\quad y(x)=\frac{\exp(-2x)-\exp(-x)}{1+\exp(-x)},\qquad \text{ii)}\quad y(x)=\frac{\ln(x)}{1+x},\ x>0,$$

$$\text{iii)}\quad y(x)=\ln(1-2x+x^2),\ x\neq 1.$$

(b) Let $y(x)=\tanh^{-1}(\sinh x)$.

(i) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cosh x}{1-\sinh^2 x}.$$

(ii) Sketch the graph of $y(x)$.

(iii) Show that, if x is small,

$$\int_0^x \sin y(t)\,\mathrm{d}t\approx x^2/2.$$

4. (a) Find

$$\text{i)}\quad \int x\sec^2 x\,\mathrm{d}x,\qquad \text{ii)}\quad \int_0^\infty x^5\exp(-x^2)\,\mathrm{d}x,\qquad \text{iii)}\quad \int_0^1 \frac{x\sin^{-1}x^2}{\sqrt{1-x^4}}\,\mathrm{d}x.$$

(b) Use the substitution $t=\tan(x/2)$ to show

$$\int_0^{\pi/2}\frac{\mathrm{d}x}{1+\cos\alpha\cos x}=\frac{\alpha}{\sin\alpha}.$$

5. (a) Find the solution of

$$\text{i)}\quad \frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x^2}=\frac{1}{x^2},\qquad \text{ii)}\quad \frac{\mathrm{d}y}{\mathrm{d}x}+y\cot x=\sin x,\quad y(0)=0.$$

(b) Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-x+1}{y+x+5}.$$

6. Solve the differential equations

(a) $y''-2y'+y=x+\exp(x)$,

(b) $x^2y''+3xy'+y=x,\quad y(1)=1/4,\quad y(2)=1$,

where a dash denotes differentiation with respect to x.

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