UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1401**

MODULE NAME : **Mathematical Methods 1**

DATE : **01-May-07**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH1401A-001-EXAM-222

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) If $\mathbf{a}$ and $\mathbf{b}$ are vectors,

   (i) define the scalar product $\mathbf{a} \cdot \mathbf{b}$,

   (ii) give a complete definition of the vector product $\mathbf{a} \wedge \mathbf{b}$.

   (b) Points with position vector $\mathbf{r}$, relative to the origin, lie in a plane with the equation $\mathbf{r}.\hat{\mathbf{n}} = c$. Show that the minimum distance of this plane from a point A is $|\mathbf{a}.\hat{\mathbf{n}} - c|$, where $\mathbf{a}$ is the position vector, relative to the orgin, of the point A.

   (c) Show that the points $P(5, 5, 3)$ and $Q(-1, 2, -3)$ lie on opposite sides of the plane $2x - 3y + 6z = 0$. Find the coordinates of the point of intersection of the line PQ and the plane.

2. (a) State De Moivre's theorem and use it to show that $(-\sqrt{3} + i)^{12} = 2^{12}$.

   (b) Find all solutions to the equation $z^3 - 1 = i\sqrt{3}$.

   (c) *You are given* that

$$1 + z + z^2 + z^3 + z^4 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

   for positive integer n. Show that

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{a^n} = \frac{a^2 - a\cos\theta}{a^2 - 2a\cos\theta + 1},$$

   where θ and a are real and $|a| > 1$. Comment on the cases $\theta = 0$ and $\theta = \pi/2$.

3. (a) Give definitions of the functions $\sinh x$, $\cosh x$, $\tanh x$ and $\coth x$. Sketch their graphs.

(b) A point P on the curve $y = \cosh x$ has coordinates $(a, \cosh a)$. The tangent to the curve at P meets the x-axis at the point T and the foot of the perpendicular from P to the x-axis is Q. Show that the length TQ is $\coth a$. Comment on the cases $a \to 0$ and $a \to \infty$.

(c) By considering the function $f(x) = (1-x)^{-1}$, show that

$$\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad |x| < 1.$$

Hence express

$$\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

as a power series in x.

4. (a) Find the integrals

i) $\displaystyle\int_0^1 \tan^{-1}(x)\,\mathrm{d}x$, ii) $\displaystyle\int_0^1 \frac{\sqrt{x}\,\mathrm{d}x}{1+x}$, iii) $\displaystyle\int 2^{-x}\tanh(2^{1-x})\,\mathrm{d}x$.

(b) Evaluate

$$\int_0^\infty \frac{\mathrm{d}x}{(x+1)^2(x^2+9)}.$$

(c) Use the substitution $x = \pi - y$ to show

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x = \frac{\pi^2}{4}.$$

5. (a) Find the solution of the following equations:

(i) $(x+1)y' + y = (x+1)^2 \sin x$,

(ii) $xy' = y + x\cot(y/x)$, $y(2) = \pi/3$,

where a prime denotes differentiation with respect to x.

(b) Use the substitution $z(x) = [y(x)]^{-3}$ to solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = (xy)^4, \quad y(1) = 1.$$

6. Solve the differential equations

(a) $y'' + y' + y = \exp(x)(1 + \sin x)$,

(b) $x^2y'' + 4xy' + 2y = 3/x^2$, $y(1) = 1$, $y'(1) = 0$,

where a prime denotes differentiation with respect to x.

END OF PAPER