1401
JAN 07

University College London
DEPARTMENT OF MATHEMATICS
Mid-sessional Examinations 2007
Mathematics 1401
Wednesday 10 January 2007 2.30 - 4.30

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

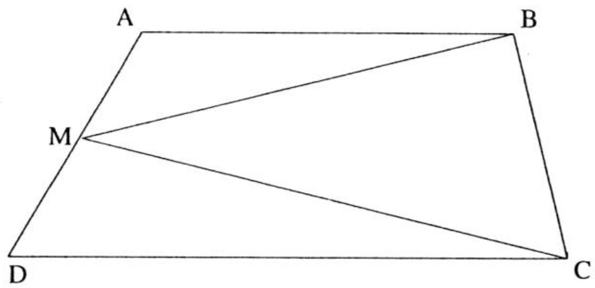
1. (a) If $\mathbf{a}$ and $\mathbf{b}$ are vectors,

   (i) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$,

   (ii) Give a careful definition of the vector product $\mathbf{a} \wedge \mathbf{b}$.

   (b) Show that the area of the triangle OAB is $\frac{1}{2}|\mathbf{a} \wedge \mathbf{b}|$, where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors relative to O of the points A and B respectively.

   (c) $ABCD$ is a trapezium (a four-sided figure with two sides parallel to one another). The lines AB and DC are parallel and M is the mid-point of AD. Show that the area of the triangle BMC is half the area of the trapezium.



2. (a) If z is a complex number then

   (i) show that if $\bar{z} = z$, with $\bar{z}$ the complex congugate of z, then z is real,

   (ii) show that for any complex number $\text{Re}(iz) = -\text{Im}(z)$,

   (iii) show that if $\text{Im}(z) > 0$ then $\text{Im}(1/z) < 0$.

   (b) Mark the positions of the roots of $z^7 = 1$ and $z^8 = 1$ on an Argand diagram. If $z^7 = 1$, $w^8 = 1$, and $z \neq w$, show that the least value that $|z - w|$ can have is $2\sin(\pi/56)$.

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3. The function defined by

$$\mathrm{gd}(t) = \tan^{-1}(\sinh t)$$

is known as the *gudermannian* of t.

(a) Show that gd is odd and sketch its graph. Sketch also the graph of $\tan^{-1} t$ on the same set of axes.

(b) Given that

$$\tan^{-1}(u) = u - \tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 - \cdots ,$$
$$\sinh t = t + \tfrac{1}{6}t^3 + \tfrac{1}{120}t^5 + \cdots ,$$

find the power series for $\mathrm{gd}(x)$ up to and including terms in x^5.

(c) Show, directly from the definition, that

$$\mathrm{gd}(t) = \int_0^t \operatorname{sech} u \, \mathrm{d}u.$$

4. (a) Find the integrals

i) $\displaystyle\int_0^\infty (x+2)\exp(6-4x-x^2)\mathrm{d}x,$ ii) $\displaystyle\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{(1-x)(x+1)}},$

iii) $\displaystyle\int_0^{\pi/2} \frac{\mathrm{d}x}{2+\cos x}.$

(b) If

$$I_n = \int t^n \exp at \, \mathrm{d}t,$$

with n a postive integer, show that

$$aI_n = t^n \exp(at) - nI_{n-1}.$$

Hence show that, given a restriction on possible values of s which you should clearly state,

$$\int_0^\infty t^n \exp(-st)\, \mathrm{d}s = n!/s^{n+1}.$$

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5. Find the solution of the equations

(a) $xy' + y = x \sin x$, $y(0) = 1$,

(b) $2xyy' = x^2 + y^2$, $y(1) = 0$,

(c) $y' + y \ln x = x^{-x}$.

where a prime denotes differentiation with respect to x.

6. Solve the differential equations

(a) $y'' - 4y = 3\exp(-2x), \quad y(0) = y'(0) = 0$,

(b) $x^2y'' - 3xy' + 4y = \cos(\ln x)$,

where a prime denotes differentiation with respect to x.

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