

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sci.*

Mathematics M14A: Mathematical Methods 1

COURSE CODE : MATHM14A

UNIT VALUE : 0.50

DATE : 11–MAY–06

TIME : 14.30

TIME ALLOWED : 2 Hours

All questions may be answered, but only marks obtained on the best **four** questions will count. The use of an electronic calculator is **not** permitted in this examination.

1. (a) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ and the vector product $\mathbf{a} \wedge \mathbf{b}$ of the two vectors \mathbf{a} and \mathbf{b} .
- (b) Show that the volume of the parallelepiped with three concurrent (i.e. sharing a common corner) edges, given by the vectors \mathbf{a} , \mathbf{b} and \mathbf{c} , is $|\mathbf{a}, \mathbf{b}, \mathbf{c}|$ where $|\mathbf{a}, \mathbf{b}, \mathbf{c}| = (\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c}$.
- (c) Show that

$$(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d}) = [\mathbf{a}, \mathbf{c}, \mathbf{d}]\mathbf{b} - [\mathbf{b}, \mathbf{c}, \mathbf{d}]\mathbf{a} = [\mathbf{d}, \mathbf{a}, \mathbf{b}]\mathbf{c} - [\mathbf{c}, \mathbf{a}, \mathbf{b}]\mathbf{d},$$

and hence find a representation of any vector in terms of three other vectors that are not themselves coplanar.

2. (a) If $z = x + iy$, express the following constraints on z as constraints on x and y .

$$\text{i) } |z - i| = |z + 1|, \quad \text{ii) } |z - i| \geq |z + i|.$$

- (b) Show that

$$1 + z + z^2 + z^3 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

for positive integer n . By considering the limit $n \rightarrow \infty$, show that if a and θ are real with $|a| < 1$ then,

$$a \sin \theta + a^2 \sin 2\theta + a^3 \sin 3\theta + \dots = \frac{a \sin \theta}{1 - 2a \cos \theta + a^2}.$$

3. (a) Show that the function

$$y(x) = \frac{\exp(ax)}{1 + \exp(x)},$$

has a stationary point at

$$x = \ln\left(\frac{a}{1-a}\right), \quad y = a^a(1-a)^{1-a},$$

for $0 < a < 1$. Sketch the graph of the function for the three cases $a = \frac{1}{2}$, $a = 1$, $a = 2$ on a single set of axes.

- (b) You are given that the function

$$y(x) = \sinh^{-1} x + (\sinh^{-1} x)^2,$$

satisfies the differential equation

$$(1 + x^2)y'' + xy' = 2,$$

where a prime denotes differentiation with respect to x . Use Leibnitz' rule for differentiating a product to show that, if n is even,

$$y^{(n)}(0) = -(-1)^{\frac{n}{2}}(n-2)^2(n-4)^2 \dots (2)^2(2),$$

where $y^{(n)}(x)$ is the n th derivative of $y(x)$. What is the equivalent expression for odd values of n ?

4. Find the integrals

$$\begin{aligned} \text{a) } & \int_0^{\infty} x \exp(-x) dx, & \text{b) } & \int_{-\infty}^1 \frac{\exp(x)}{1 + 2 \exp(x)} dx, & \text{c) } & \int_0^1 x \tan^{-1}(x) dx, \\ \text{d) } & \int \frac{\tan(\ln x)}{x} dx, & \text{e) } & \int_1^{\infty} \frac{dx}{x^3 + x^2 + x}. \end{aligned}$$

5. Find the solution of the equations

- (a) $y' = 2 + y^2$, $y(0) = 0$.
 (b) $y' + y \sin x = x \exp(\cos x)$, $y(0) = 0$,
 (c) $4(2x^2 + xy)y' = 3y^2 + 4xy$.

where a prime denotes differentiation with respect to x .

6. Solve the differential equations

- (a) $x^2 y'' - 2xy' + 2y = 1$, $y(1) = 1$, $y'(1) = 0$,
 (b) $y'' + y' - 6y = x + \exp(2x)$, $y(0) = 0$, $y'(0) = 1/5$.

where a prime denotes differentiation with respect to x .