1401 2006

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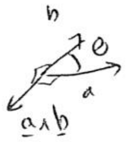
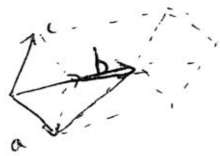
MAY 06
SOL^{N}S

1 a) $\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\theta$ where θ is the angle between the vectors $\underline{a}$ & $\underline{b}$



$\underline{a}\wedge\underline{b} = |\underline{a}||\underline{b}|\sin\theta\ \hat{\underline{n}}$ where $\hat{\underline{n}}$ is a unit vector, normal to both $\underline{a}$ & $\underline{b}$ & chosen so that $\underline{a}, \underline{b}, \underline{a}\wedge\underline{b}$ form a right-handed set.

b) Volume is (Area of base) × perpendicular ht.)



Choose the parallelogram with sides $\underline{a}, \underline{b}$ as "base"

Area of base is $2 \cdot \frac{1}{2}|\underline{a}\wedge\underline{b}| = |\underline{a}\wedge\underline{b}|$

Height is $\underline{c}\cdot\hat{\underline{n}}$ with $\hat{\underline{n}}$ a vector normal to $\underline{a}$ & $\underline{b}$ eg $\dfrac{\underline{a}\wedge\underline{b}}{|\underline{a}\wedge\underline{b}|}$

$\Rightarrow$ Volume is $\left(\dfrac{\underline{a}\wedge\underline{b}}{|\underline{a}\wedge\underline{b}|}\right)\cdot\underline{c} \times |\underline{a}\wedge\underline{b}| = (\underline{a}\wedge\underline{b})\cdot\underline{c}$. If we had taken the normal vector to point in the direction opposite to this then this final answer would be negative. Hence the || signs in the final answer

c) We know $(\underline{p}\wedge\underline{q})\wedge\underline{r} = \underline{q}(\underline{p}\cdot\underline{r}) - \underline{p}(\underline{q}\cdot\underline{r})$

So $(\underline{a}\wedge\underline{b})\wedge(\underline{c}\wedge\underline{d}) = \underline{b}(\underline{a}\cdot(\underline{c}\wedge\underline{d})) - \underline{a}(\underline{b}\cdot(\underline{c}\wedge\underline{d}))$

$= [\underline{a},\underline{c},\underline{d}]\underline{b} - [\underline{b},\underline{c},\underline{d}]\underline{a}$

(the $\cdot$ & $\wedge$ may be interchanged)

Alternatively it is $= \underline{c}((\underline{a}\wedge\underline{b})\cdot\underline{d}) - \underline{d}((\underline{a}\wedge\underline{b})\cdot\underline{c})$

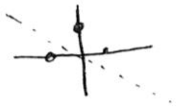
$= [\underline{d},\underline{a},\underline{b}]\underline{c} - [\underline{c},\underline{a},\underline{b}]\underline{d}$

Equating these we can write

$$\underline{d} = \frac{[\underline{b},\underline{c},\underline{d}]\underline{a} + \underline{b}[\underline{a},\underline{d},\underline{c}] + \underline{c}[\underline{a},\underline{b},\underline{d}]}{[\underline{a},\underline{b},\underline{c}]}$$

unless $[\underline{a},\underline{b},\underline{c}] = 0$ ie $\underline{a},\underline{b},\underline{c}$ are coplanar.

2 a) $z = x + iy$ i), $|z - i| = |z + 1|$ $\Rightarrow$ distance from i = distance from -1



$\Rightarrow$ $\underline{y = -x}$

ii) $|z - i| \geqslant |z + i|$ $\Rightarrow$ distance from i $\geqslant$ distance from $-i$

$\Rightarrow$ $\underline{x \leq 0}$.

b) $S_n = 1 + z + \dots z^n$ $\Rightarrow$ $(1-z)S_n = 1 - z^{n+1}$

$zS_n = z \dots \dots z^n + z^{n+1}$

$$S_n = \frac{1 - z^{n+1}}{1 - z}$$

If $|z| < 1$ then as $n \to \infty$ we get $S_n = 1/1-z$

Choosing $z = ae^{i\theta}$ & taking the imaginary part & using $\mathrm{Im}[e^{in\theta}] = \sin n\theta$ we get

$$a \sin\theta + a^2 \sin 2\theta + \dots = \mathrm{Im}\, \frac{1}{1 - ae^{i\theta}}$$

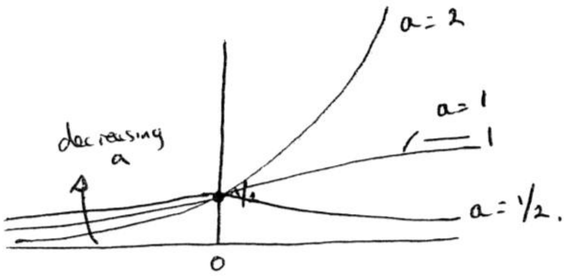
$$= \mathrm{Im}\, \frac{1 - ae^{-i\theta}}{1 - ae^{i\theta} - ae^{-i\theta} + a^2}$$

$$= \frac{a \sin\theta}{1 - 2a\cos\theta + a^2}.$$

3 a) $y(x) = \dfrac{e^{ax}}{1+e^x}$, $y'(x) = \dfrac{a\,e^{ax}}{(\ )} - \dfrac{(e^x)e^{ax}}{(\ )^2}$

$y' = 0$ if $(1+e^{x})a e^{ax} = e^x e^{ax} \Rightarrow e^x = \dfrac{a}{1-a}$

$\Rightarrow y = \dfrac{a^a(1-a)^{-a}}{1/1-a} = \underline{a^a(1-a)^{1-a}}$; $\underline{x = \ln(a/1-a)}$



As $x \to \infty$ $y \sim e^{(a-1)x}$

As $x \to -\infty$ $y \sim e^{ax}$

At $x = 0$ $y = 1/2$

We have a turning point only if $a = 1/2$ at $x = \ln(1/2 / 1/2) = 0$, $y = \left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{2}\right)^{1/2} = 1/2$.

The case $a = 1/2$ is an even function as $\dfrac{e^{1/2 x}}{1+e^x} = \dfrac{1}{e^{-x/2}+e^{x/2}}$

$= \frac{1}{2}\operatorname{sech} x/2$.

b) $(1+x^2)y'' + xy' = 2 \quad * \Rightarrow (1+x^2)y^{n+2} + n.2x.y^{n+1} + \dfrac{n(n-1)}{2}.2y^n$

$+ xy^{n+1} + ny^n = 0$

Putting $x = 0$ gives $y^{n+2}(0) = -n^2 y^n(0)$

So $y^4(0) = -2^2y^2(0)$ & $y^2 = 2$, putting $x = 2$ in $*$

$= -4.2$

$y^6(0) = -4^2(-2^2.2) = 4^2 2^2.2$ & we see $\underline{y^n = \dfrac{(-1)(-1)^{n/2}(n\cdot\ )^2(n-4)^2\ldots}{2^2.2}.}$

If n is odd we need $y'(0) = \dfrac{1}{\sqrt{1+x^2}} + \dfrac{2\sinh^{-1}x}{\sqrt{1+x^2}}\Bigg|_{x=0} = 1$

& $y^3(0) = -1^2.1$, $y^5(0) = -3^2(-1^2.1) = 3^2.1^2.1$

& $\underline{y^n = (n-2)^2(n-4)^2 \ldots 3^2.1^2.1.(-1)^{\frac{n}{\ }}}$

4) a) $\displaystyle\int_0^\infty x e^{-x}\,dx = \Big[-e^{-x}\cdot x\Big]_0^\infty + \int_0^\infty e^{-x}\,dx = \underline{1}$

b) $\displaystyle\int_{-\infty}^{1} \frac{e^x}{1+2e^x}\,dx = \left[\frac{1}{2}\ln(1+2e^x)\right]_{-\infty}^{1} = \underline{\frac{1}{2}\ln(1+2e)}$

c) $\displaystyle\int_0^1 x\tan^{-1}x\,dx = \left[\frac{x^2}{2}\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x^2}{2}\frac{1}{1+x^2}\,dx$

$\displaystyle = \frac{\pi}{8} - \frac{1}{2}\int_0^1 1 - \frac{1}{1+x^2}\,dx = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \underline{\frac{\pi}{4} - \frac{1}{2}}.$

d) $\displaystyle\int \frac{\tan(\ln x)}{x}\,dx = \underline{-\ln(\cos(\ln x)) + C}.$

e) $\displaystyle\int_1^\infty \frac{dx}{x^3+x^2+x} = \int_1^\infty \frac{A}{x} + \frac{Bx+C}{x^2+x+1}\,dx$ where

$A+B=0$
$A+C=0$
$A=1$
$B=C=-1$

$\displaystyle = \int_1^\infty \frac{1}{x} - \frac{1}{2}\frac{2x+1}{x^2+x+1} - \frac{1}{2}\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$

$\displaystyle = \left[\ln\frac{x}{\sqrt{x^2+x+1}} - \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\frac{(x+\frac{1}{2})2}{\sqrt{3}}\right]_1^\infty$

$\displaystyle = 0 - \frac{1}{\sqrt{3}}\frac{\pi}{2} - \ln\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3} = \underline{\ln\sqrt{3} - \frac{\pi}{6\sqrt{3}}}.$

5 a) $y' = 2 + y^2 \Rightarrow \int \frac{dy}{2+y^2} = \int dx \Rightarrow \frac{1}{\sqrt{2}} \tan^{-1} \frac{y}{\sqrt{2}} = x + C$

$y(0) = 0 \Rightarrow C = 0 \Rightarrow y = \sqrt{2} \tan \sqrt{2}x$.

b) $y' + y \sin x = x e^{\cos x} \Rightarrow \left[y e^{-\cos x} \right]' = x$

$e^{\int \sin x \, dx} = e^{-\cos x} \Rightarrow y = \left(x^2/2 + A \right) e^{\cos x}$

$y(0) = 0 \Rightarrow A = 0, \quad y = \frac{1}{2} x^2 e^{\cos x}$

c) $4(2x^2 + xy) \frac{dy}{dx} = (3y^2 + 4xy)$

Put $y = xv \Rightarrow 4(2x^2 + x^2 v)(xv' + v) = 3x^2 v^2 + 4x^2 v$

$\Rightarrow xv' + v = \frac{3v^2 + 4v}{8 + 4v} \Rightarrow xv' = \frac{3v^2 + 4v - 8v - 4v^2}{8 + 4v}$

$\Rightarrow \int \frac{8 + 4v}{v^2 + 4v} dv = 2 \int \frac{(2v + 4)}{v^2 + 4v} dv = -\int \frac{dx}{x}$

$\Rightarrow 2 \ln(v^2 + 4v) + \ln x = C$

$\Rightarrow \left(\frac{y^2}{x^2} + \frac{4y}{x} \right)^2 \cdot x = A \Rightarrow \underline{(y^2 + 4xy)^2 = Ax^3}$

6 b) $y'' + y' - 6y = x + e^{2x}$, $y(0) = 0$, $y'(0) = 1/5$

ae $\alpha^2 + \alpha - 6$: $(\alpha+3)(\alpha-2) = 0$ $\Rightarrow$ CF $y = Ae^{-3x} + Be^{2x}$

For PI try $y = ax + b + cxe^{2x}$ & substitute

$$\left(2c.2e^{2x} + \cancel{4cxe^{2x}}\right) + \left(a + ce^{2x} + \cancel{2cxe^{2x}}\right) - 6\left(ax + b + \cancel{cxe^{2x}}\right) = x + e^{2x}$$

$\Rightarrow$ $c = 1/5$, $a = -1/6$, $b = -\frac{1}{36}$

$\Rightarrow$ $y = Ae^{-3x} + Be^{2x} - x/6 - 1/36 + \frac{1}{5}xe^{2x}$

$y(0) = 0$ $\Rightarrow$ $0 = A + B - 1/36$ $\qquad$ $A + B = 1/36$

$y'(0) = 1/5$ $\Rightarrow$ $1/5 = -3A + 2B - 1/6 + \frac{1}{5}$ $\qquad$ $3A - 2B = -1/6$

$5A = -\frac{1}{6} + \frac{2}{36} = \frac{-6+2}{36} = -\frac{4}{36}$

ie $A = \frac{-4}{180}$, $B = \frac{5}{180} - \frac{-4}{180} = \frac{9}{180}$

$$\underline{y = \frac{1}{180}\left(9e^{2x} - 4e^{-3x}\right) - \frac{1}{6}\left(x + 1/6\right) + \frac{1}{5}xe^{2x}}.$$

a) $x^2y'' - 2xy' + 2y = 1$, $y(1) = 1$, $y'(1) = 0$

If $x = e^t$, then $y_{tt} - 3y_t + 2y = 1$, a.e. $\alpha^2 - 3\alpha + 2 = (\alpha-2)(\alpha-1) =$

$\Rightarrow$ $y(t) = Ae^{2t} + Be^{t} + 1/2$ $\Rightarrow$ $y(x) = Ax^2 + Bx + 1/2$

$y(1) = 1 \Rightarrow$ $A + B = 1/2$

$y'(1) = 0$ $\quad$ $2A + B = 0$ $\Rightarrow$ $A = -1/2$, $B = 1$ $\Rightarrow$ $\underline{y = -\frac{x^2}{2} + x + 1/2}$