University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2006
Mathematics M14A
Tuesday 10 January 2006 3.30 - 5.30 or 4.05 - 6.05

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a}.\mathbf{b}$ and vector product $\mathbf{a} \wedge \mathbf{b}$ of the two vectors $\mathbf{a}$ and $\mathbf{b}$.

   (b) Show that the area of the triangle OAB is $\frac{1}{2}|\mathbf{a} \wedge \mathbf{b}|$, where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors relative to O of the points A and B respectively.

   (c) Three non-collinear points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively relative to an origin O, which is not necessarily in the plane PQR. Show that the area of the triangle PQR is equal to the magnitude of the vector

$$\tfrac{1}{2}\left(\mathbf{q} \wedge \mathbf{r} + \mathbf{r} \wedge \mathbf{p} + \mathbf{p} \wedge \mathbf{q}\right).$$

   When O does lie in the plane PQR and inside the triangle PQR, interpret this result in terms of the areas of the triangles OQR, ORP and OPQ.

2. (a) Find the least value of $|z|$ if $|z-1| = |z-4+i|$.

   (b) Write down expressions for $\cos\theta$ and $\sin\theta$ in terms of $\exp(i\theta)$ and show that

$$1 + \exp(\theta) = 2\exp(\tfrac{1}{2}i\theta)\cos\tfrac{1}{2}\theta.$$

   Hence show that

$$1 + 4\cos\theta + 6\cos 2\theta + 4\cos 3\theta + \cos 4\theta = 16\cos 2\theta\left(\cos\tfrac{1}{2}\theta\right)^4.$$

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3. (a) Define $\sinh x$ and $\cosh x$ in terms of the exponential function. Use these definitions to find expressions for their derivatives and to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

(b) Draw graphs of the functions $\cos x$ and $\tan x$ for x in the range $-2\pi < x < 2\pi$, on the same set of axes. Show that where the curves meet they do so at right angles.

(c) Use the binomial expansion to show

$$\frac{1}{\sqrt{1-x^2}} = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \cdots,$$

and hence find the Maclaurin series for $\sin^{-1} x$ correct to the term in x^7. Show that the general term in the series is $(2n)!x^{2n+1}/(4^n(n!)^2(2n+1))$.

4. (a) Find the integrals

$$\int_0^\infty \frac{x\,\mathrm{d}x}{(x+2)(x^2+4)}, \quad \int \sqrt{x}\exp(\sqrt{x})\,\mathrm{d}x, \quad \int_{\pi/4}^{\pi/2} \cot^2 x\,\mathrm{d}x.$$

(b) If n and m are non negative integers and

$$I_{n,m} = \int_0^1 (\ln x)^n\, x^m\,\mathrm{d}x,$$

show that

$$(m+1)I_{n,m} + nI_{n-1,m} = 0,$$

and hence that $I_{n,m} = (-1)^n n!/(m+1)^{n+1}$.

5. Find the solution of the equations

(a) $(x-1)y' = (y+1)$, $y(0) = 1$,

(b) $(x+1)y' - 3y = (x+1)^5$, $y(0) = 1$,

(c) $(x-y+1)y' = (x-y-1)$,

where a prime denotes differentiation with respect to x.

6. Solve the differential equations

(a) $y'' - 5y' + 4y = \cosh(x)$, $\quad y(0) = y'(0) = 0$,

(b) $x^2y'' + xy' - 9y = x^3$,

where a prime denotes differentiation with respect to x.

END OF PAPER