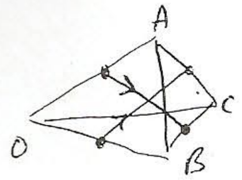
1401 M14A Solutions S2005

$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$, θ the angle between $\underline{a}$ & $\underline{b}$

$\underline{a}\wedge\underline{b} = |\underline{a}||\underline{b}|\sin\theta\,\hat{\underline{c}}$ where $\hat{\underline{c}}$ normal to $\underline{a}$ & $\underline{b}$ & such that $\underline{a}, \underline{b}, \underline{a}\wedge\underline{b}$ forms a right-handed set. (soon)

b) 

Mid pt of OA is $\underline{a}/2$

" of BC is $\frac{1}{2}(\underline{b}+\underline{c})$

line is $\frac{1}{2}(\underline{b}+\underline{c}-\underline{a})$

A similar vector from OB to AC is $\frac{1}{2}(\underline{a}+\underline{c}-\underline{b})$

Lines meet if there is solution to $\frac{1}{2}\underline{b} + \lambda\frac{1}{2}(\underline{a}+\underline{c}-\underline{b}) = \frac{1}{2}\underline{a} + \frac{\mu}{2}(\underline{b}+\underline{c}-\underline{a})$

which is satisfied if $\lambda = \mu = 1/2$. Third line will meet at same point, $\frac{1}{4}(\underline{a}+\underline{b}+\underline{c})$ by symmetry

Angle between lines is $90°$ if

$$(\underline{b}+\underline{c}-\underline{a}).(\underline{a}+\underline{c}-\underline{b}) = 0$$

but ie $\underline{a}.\underline{b} + \underline{b}.\underline{c} - \underline{b}.\underline{b} + \underline{a}.\underline{c} + \underline{c}.\underline{c} - \underline{c}.\underline{b} - \underline{a}.\underline{a} - \underline{a}.\underline{c} + \underline{a}.\underline{b}$

$= 2\underline{a}.\underline{b} - \underline{a}^2$

$= a^2 + b^2 - (\underline{a}-\underline{b})^2 - a^2 = b^2 - (\underline{a}-\underline{b})^2 = 0$ as $|\underline{b}| = |\underline{a}-\underline{b}|$

better is $(\underline{c}+(\underline{b}-\underline{a})).(\underline{c}-(\underline{b}-\underline{a})) = \underline{c}^2 - (\underline{b}-\underline{a})^2$

$= 0$ as $|\underline{c}| = |\underline{b}-\underline{a}|$

2) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

a) $(1+i\sqrt{3})^{-3} = \left[2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]^{-3} = \left[2\left(\cos\pi/3 + i\sin\pi/3\right)\right]^{-3}$

$= \frac{1}{8}\left[\cos\pi - i\sin\pi\right] = -1/8$ & hence result

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

b) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta = c^4 + 4ic^3s + 6(-1)c^2s^2 + 4(-i)cs^3 + s^4$

$\Rightarrow \cos 4\theta = c^4 - 6c^2s^2 + s^4$, $\sin 4\theta = 4c^3s - 4cs^3$

Take ratio, divide by c^4 & get

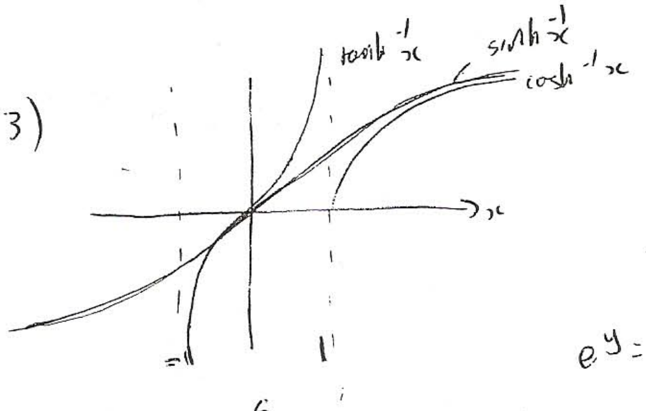
$\tan 4\theta = (4t - 4t^3)/1 - 6t^2 + t^4$, $t = \tan\theta$.

If $x = \tan\theta$ then we have roots if $\tan 4\theta = 1$

ie $\quad 4\theta = \frac{\pi}{4} + n\pi \quad \Rightarrow \tan\theta = x = \tan\left(\pi/16 + \frac{n\pi}{4}\right)$,

$n = 0, 1, 2, 3$ pick out different roots.

3)



If $y = \sinh^{-1}x$ then

$x = \frac{1}{2}(e^y - e^{-y})$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0$

$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$

$\underline{6}$

need + root as $e^y > 0$ $\qquad y = \ln(x + \sqrt{1+x^2})$

If $(1+x^2)y'' + xy' = 2$ then

$(1+x^2)y^{n+2} + 2nxy^{n+1} + 2n(n-1)/2 \cdot y^n + xy^{n+1} + ny^n = 0$

& at $x = 0$ $\qquad y^{n+2} + n^2 y^n = 0.$

Now $\quad y(0) = \ln 1 = 0 \quad , \quad y'(0) = \frac{1 + x/\sqrt{1+x^2}}{x + \sqrt{1+x^2}} \cdot (\sinh^{-1}x) \Big|_0 = 0$

$y''(0) = 2$ from ode.

Therefore: $\quad y^3 = y^5 = y^7 = \cdots = 0 \quad , \quad y^4 = -2^2 y^2 = -8 \quad , \quad y^6 = -4^2.(-8) = 4.4.8$

$y^8 = -36.4.4.8$

& $\quad (\sinh^{-1}x)^2 = \frac{2.x^2}{2} + \frac{(-8)x^4}{24} + \frac{4.\cancel{4}.\cancel{8}x^6}{\cancel{6}.5.\cancel{4}.3.\cancel{2}} - \frac{\cancel{8}.\cancel{4}.\cancel{4}.\cancel{6}.\cancel{6}}{\cancel{8}.7.\cancel{6}.5.\cancel{4}.\cancel{3}.\cancel{2}}x^8 \ldots$

$= x^2 - \frac{x^4}{3} + \frac{8}{45}x^6 - \frac{4}{35}x^8 \ldots$

4 a) $\dfrac{1}{\cos\theta\sin\theta} = \dfrac{1/\cos^2\theta}{\sin\theta/\cos\theta} = \dfrac{\sec^2\theta}{\tan\theta}$.

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin x/2 \cos x/2} = \frac{1}{2}\int \frac{\sec^2 x/2}{\tan x/2} = \ln\left(\tan\frac{x}{2}\right) + C$$

$\dfrac{dy}{dx} + \dfrac{1}{\sin x}y = \sin x$. IF is $e^{\int \frac{1}{\sin x}dx} = e^{\ln\tan x/2} = \tan x/2$.

$\Rightarrow \dfrac{d}{dx}\left[y\tan x/2\right] = 2\sin x/2 \cdot \cos x/2 \cdot \dfrac{\sin x/2}{\cos x/2} = 2\sin^2 x/2$

$\Rightarrow y = \dfrac{1}{\tan x/2}\left\{A + x - \sin x\right\}$

Finite at $x=0 \Rightarrow A=0$ & $y = \dfrac{x - \sin x}{\tan x/2}$

b) $I = \displaystyle\int_{\pi/2}^{\pi} \frac{d\theta}{1+\sin\theta - \cos\theta} = \int_1^\infty \frac{1}{1+\frac{2t}{1+t^2} - \frac{(1-t^2)}{1+t^2}}\frac{2dt}{1+t^2} = \int_1^\infty \frac{dt}{t^2+t}$

$t = \tan\theta/2$, $dt = \frac{1}{2}\sec^2\theta/2\, d\theta = (1+t^2)\,d\theta/2$

$\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$

$= \displaystyle\int_1^\infty \frac{1}{t} - \frac{1}{1+t}\,dt$

$= \left[\ln\left\{\dfrac{t}{1+t}\right\}\right]_1^\infty$

$= -\ln\frac{1}{2} = \underline{\ln 2}$.

5 a) Put $y - x = v(x)$ so $\dfrac{dy}{dx} = 1 + \dfrac{dv}{dx}$ & $1 + \dfrac{dv}{dx} = \dfrac{v+2}{v-4}$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{6}{v-4} \Rightarrow \dfrac{v^2}{2} - 4v = 6x + C$, $\underline{(y-x)^2 - 8(y-x) = 12x + C}$

b) Put $y = vx$ then $x^2 v(xv' + \cancel{v}) = x^2\cancel{v^2} - \sqrt{x^2 + x^2v^2}$

$\Rightarrow x^3 v\dfrac{dv}{dx} = -x\sqrt{1+v^2}$ & $\displaystyle\int \frac{v\,dv}{\sqrt{1+v^2}} = \int -\frac{dx}{x^2}$

$\Rightarrow \sqrt{1+v^2} = C + 1/x$ when $x = 1$; $v = y/x = 0$:
$\Rightarrow C = 0$

& $1 + y^2/x^2 = \dfrac{1}{x^2} \Rightarrow \underline{y^2 = 1 - x^2}$ (This substitution seen in other contexts)

6 a) $y'' - 7y' + 6y = \frac{1}{2}(e^{x} + e^{-x}) + x$

$\alpha^2 - 7\alpha + 6 = 0$, $(\alpha - 6)(\alpha - 1) = 0$

CF. $y = Ae^{6x} + Be^{x}$ for PI try $\underline{y = Ce^{-x} + Dxe^{x} + Ex + F}$

with: $[Ce^{-x} + \cancel{Dxe^{x}} + 2De^{x}] - 7[-Ce^{-x} + De^{x} + \cancel{Dxe^{x}} + E]$

$+ 6[Ce^{-x} + \cancel{Dxe^{x}} + Ex + F] = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x} + x$

$\Rightarrow \underline{C = \frac{1}{28}, \quad D = -\frac{1}{10}, \quad E = \frac{1}{6}, \quad F = 7/36}.$

b) If $x = e^{t}$,

$\ddot{y} + 3\dot{y} + 2y = 3e^{-2t}$

a.e. $\alpha^2 + 3\alpha + 2 = 0$ $(\alpha+2)(\alpha+1) = 0$, $y = Ae^{-2t} + Be^{-t}$

& PI : $y = Cte^{-2t}$ where $(4C\cancel{te^{-2t}} - 4Ce^{-2t}) + 3(-2\cancel{tCe^{-2t}}$
$+ Ce^{-2t}) + 2\cancel{te^{-2t}} = 3e^{-2t}$

so $C = -3$

$y = \frac{A}{x^2} + \frac{B}{x} - 3\frac{\ln x}{x^2}$

$\left.\begin{array}{ll} y(1) = 1 \Rightarrow & A + B = 1 \\ y'(1) = 1 \Rightarrow & -2A - B - 3 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} A = -5 \\ B = 6 \end{array}$, $\underline{y = \frac{6}{x} - \frac{5}{x^2} - \frac{3\ln x}{x^2}}$