M14A/1401

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JAN 05

University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2005
Mathematics M14A
Friday 14 January 2005 2.30 - 4.30

All questions may be attempted but only the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the scalar product $\mathbf{a}.\mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$.

   (b) Two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ have an angle γ between them. Show that the vector $\hat{\mathbf{b}} - (\hat{\mathbf{a}}.\hat{\mathbf{b}})\hat{\mathbf{a}}$ is perpendicular to $\hat{\mathbf{a}}$ and of length $|\sin\gamma|$.

   (c) In a triangle ABC the perpendiculars from B and C to the opposite sides meet at D. Prove that $\vec{\mathbf{AC}}.(\vec{\mathbf{DA}} + \vec{\mathbf{AB}}) = \vec{\mathbf{AB}}.(\vec{\mathbf{DA}} + \vec{\mathbf{AC}}) = 0$, and hence that $\vec{\mathbf{DA}}.\vec{\mathbf{BC}} = 0$, showing that the perpendiculars from all the vertices to opposite sides meet at D.

2. (a) Use that fact that $\mathrm{Re}(\exp(iax)) = \cos ax$ to find

$$\int_0^\infty \exp(bx)\cos ax\, dx,$$

   where a and b are real. For what values of b does the integral exist?

   (b) If $z = \cos\theta + i\sin\theta$ show, by expanding $(z+z^{-1})^5(z-z^{-1})^5$, that

$$\sin^5\theta\cos^5\theta = \frac{1}{2^9}(\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta).$$

   Show

$$\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^5\theta\, d\theta = \frac{1}{60}.$$

PLEASE TURN OVER

3. (a) Define $\cosh x$ and $\sinh x$. Show that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}.$$

Deduce

$$\int_0^1 \frac{u}{\sqrt{1+u^4}}\mathrm{d}u = \frac{1}{2}\ln(1+\sqrt{2}).$$

(b) Show

$$\int_0^\infty \frac{x}{(x+1)(x^2+4)}\mathrm{d}x = \frac{1}{5}(\pi - \ln 2), \qquad \int_{\exp(\pi/4)}^{\exp(\pi/3)} \frac{\cot(\ln x)}{x}\mathrm{d}x = \frac{1}{2}\ln\left(\frac{3}{2}\right).$$

4. (a) If

$$f(x) = \frac{\exp(2x) - 2\exp(x) + 1}{\cos 3x - 2\cos 2x + \cos x},$$

use series expansions to show that $\lim_{x\to 0} f(x) = \lim_{x\to 0} f'(x) = -1$. What is $\lim_{x\to 0} f''(x)$?

(b) Given p and n are non-negative integers and

$$I_{p,n} = \int_0^1 (1-x)^p x^n \,\mathrm{d}x, \quad p \geq 0,\ n \geq 0,$$

prove that, for p positive,

$$(n+1)I_{p,n} = pI_{p-1,n+1},$$

and also that

$$(p+n+1)I_{p,n} = pI_{p-1,n}.$$

Hence or otherwise show that, if p and n are postive integers,

$$I_{p,n} = \frac{p!n!}{(p+n+1)!}.$$

5. Solve the differential equations

$$\text{(a)} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x-1} = \exp(x),$$

$$\text{(b)} \qquad x\frac{\mathrm{d}y}{\mathrm{d}x} = y + x\tan\left(\frac{y}{x}\right), \quad y(1/2) = \pi/4.$$

6. Find the general solutions of

$$\text{(a)} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin x + \sin^2 x,$$

$$\text{(b)} \qquad x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2.$$

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