UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M14A: Mathematical Methods 1

COURSE CODE	:	**MATHM14A**
UNIT VALUE	:	**0.50**
DATE	:	**13-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0953-3-210

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) If **a**, **b** and **c** are three vectors.
   (i) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$,
   (ii) Give a careful definition of the vector product $\mathbf{a} \wedge \mathbf{b}$,
   (iii) Define the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

   (b) Show that a plane containing the origin, the x-axis and a vector with direction cosines (p, q, r) has Cartesian equation.
   $$ry = qz.$$

   (c) Find the acute angle between the line $\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ and the plane $\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 10$. How far is the plane from the point $(1, 0, 0)$?

2. (a) If z is the complex number $x + iy$, show that
   $$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

   (b) Recall that $\exp(i\theta) = \cos\theta + i\sin\theta$ for real values of θ. Show
   $$\frac{1}{1 + \exp(i\theta)} = \frac{1}{2} - \frac{i}{2}\tan(\theta/2)$$
   and describe the path followed by the point $w = 1/(1 + z)$ as z starts at $z = 1$ and moves, in an anticlockwise direction, around the unit circle in the Argand diagram.

   (c) Use complex numbers to evaluate the integral
   $$\int_0^{\pi} \exp(-ax)\sin(nx)\, dx,$$
   for integer valuies of n. Comment on the limit $a \to \infty$.

3. (a) Write down the Maclaurin expansions (i.e. about $x = 0$) of $\sin x$ and $\cos x$ clearly giving the general term in the expansions.

(b) Consider the function

$$f(x) = \frac{\sin x}{b + \cos ax}.$$

(i) Show the nth derivative $f^{(n)}(0) = 0$ if n is an even integer,

(ii) Find a relation between a and b that ensures $f^{(3)}(0) = 0$.

(c) The function $y(x)$ satisfies the equations

$$\frac{d^2y}{dx^2} - xy = 0, \quad y(0) = 1, \quad y^{(1)}(0) = 0.$$

Use Leibniz theorem for differentiating a product to show that

$$y^{(n)}(0) = (n-2)y^{(n-3)}(0), \quad n > 2.$$

Hence write down the first 5 non-zero terms in the Maclaurin expansion of $y(x)$ about $x = 0$.

4. (a) Evaluate the following integrals.

$$\int_0^\infty \frac{2\,dx}{(x+1)(x^2+1)}\,; \qquad \int_0^a \sinh^{-1} x\,dx.$$

(b) If $t = \tanh x$, show that

$$\cosh^2 x = \frac{1}{1-t^2} \quad \text{and} \quad \sinh^2 x = \frac{t^2}{1-t^2}.$$

Use the substitution $t = \tanh x$ to evaluate

$$\int \frac{dx}{1 + a\sinh^2 x}, \quad a > 1.$$

5. (a) Solve the equations

(*i*) $x\dfrac{dy}{dx} - 2y = x^3 \ln x, \quad y(1) = -1,$ (*ii*) $x^2\dfrac{dy}{dx} = xy + y^2, \quad y(1) = 1.$

(b) Use the transformation $u = y^{-2}$ to solve the equation

$$x\frac{dy}{dx} + 2y = x^3y^3.$$

6. Solve the differential equations

(a) $y'' - 4y' + 4y = \cosh ax, \quad a \neq 2,$

(b) $x^2y'' + xy' + 4y = \ln x, \quad x > 0, \quad y(1) = y'(1) = 1$

where a dash ($'$) denotes differentiation with respect to x.

END OF PAPER