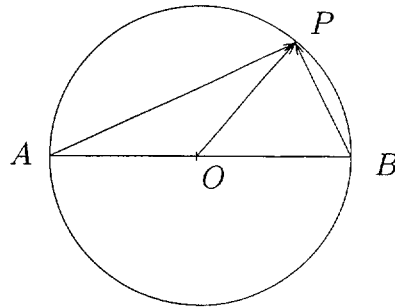
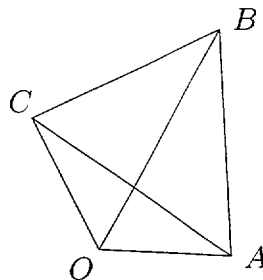


All questions may be answered, but only marks obtained on the best **four** questions will count. The use of an electronic calculator is **not** permitted in this examination.

1. (a) Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ of the vectors \mathbf{a} and \mathbf{b} . How can the scalar product be used to test if two vectors are normal to one another? Geometrically, when is $(\mathbf{a} \cdot \mathbf{b})^2 = a^2b^2$?



- (b) The diagram shows a circle radius a and centre O . Show that $\vec{AP} \cdot \vec{BP} = 0$ and deduce that the angle APB is a right angle.
2. (a) Show that the area of the triangle AOB is $\frac{1}{2}|\mathbf{a} \wedge \mathbf{b}|$, where \mathbf{a} and \mathbf{b} are the position vectors, relative to O , of A and B respectively.
- (b) Three points A ; B and C with position vectors \mathbf{a} , \mathbf{b} and \mathbf{c} relative to O define a triangle ABC . Show that the area of the triangle is $\frac{1}{2}|\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a}|$.
- (c) In the diagram $OABC$ is a quadrilateral such that the diagonal OB bisects the diagonal AC .



Show $\mathbf{a} + \mathbf{c} = 2t\mathbf{b}$, for some scalar t where \mathbf{a} , \mathbf{b} and \mathbf{c} are the position vectors of points A , B and C relative to O . Hence show that the line OB cuts the quadrilateral into two parts with equal area.

3. (a) State De Moivre's theorem and use it to evaluate

$$(1 + \sqrt{3}i)^{10} - (1 - \sqrt{3}i)^{10}.$$

- (b) Find the n solutions of $z^n = 1$ with n a positive integer and plot them in the Argand diagram. Show that their sum is zero and hence deduce that

$$\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}.$$

4. (a) Define $\sinh x$, $\cosh x$ in terms of the exponential function. Use these definitions to show that $\frac{d}{dx} \cosh x = \sinh x$ and $\cosh^2 x - \sinh^2 x = 1$.

- (b) If

$$I_n = \int \cosh^n x \, dx, \quad n = 0, 1, 2, \dots,$$

show that

$$nI_n = \sinh x \cosh^{n-1} x + (n-1)I_{n-2}, \quad n \geq 2,$$

and hence or otherwise evaluate

$$\int_0^x \cosh^4 u \, du.$$

5. (a) Write down the Taylor series expansion of the functions $\sin x$ and $\exp x$ about $x = 0$, giving the general term in each expansion.

Use these to show that

$$\exp(\sin x) = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{15}x^5 + \dots.$$

Hence write down the expansions of

$$\frac{1}{\exp(\sin x)} \quad \text{and} \quad \sinh(\sin x)$$

correct to terms in x^5 .

- (b) Evaluate the integrals

$$\int \frac{x^2 + 1}{x(x-4)^2} \, dx, \quad \int \frac{\cot(\ln x)}{x} \, dx.$$

6. (a) Show that the substitution $z(x) = (y(x))^{1-n}$, replacing the dependent variable $y(x)$ by $z(x)$, reduces the nonlinear equation

$$\frac{dy}{dx} + a(x)y = f(x)y^n$$

to the linear equation

$$\frac{dz}{dx} + (1-n)a(x)z = (1-n)f(x).$$

Hence show that the solution to

$$x \frac{dy}{dx} + y = x^4 y^3, \quad y(1) = 1$$

is

$$y = \frac{1}{\sqrt{2x^2 - x^4}}.$$

- (b) Solve the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 4y = 16, \quad y(1) = \frac{dy}{dx}(1) = 0.$$