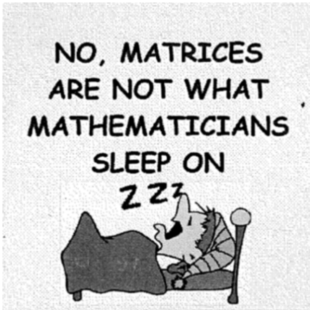
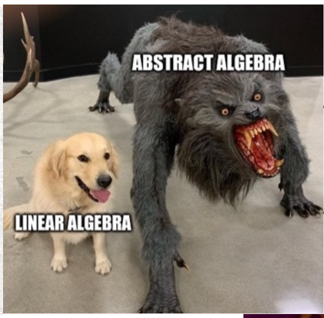
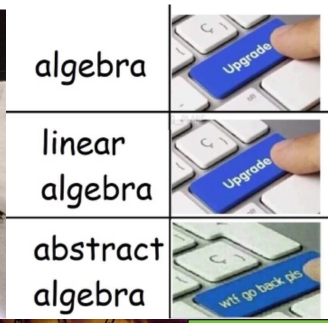
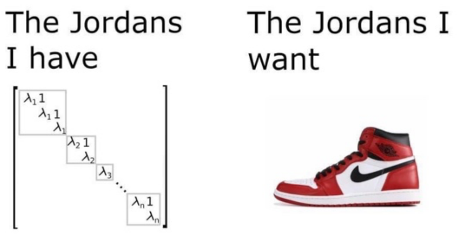
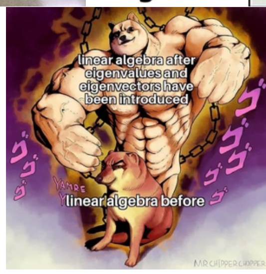
Linear Algebra: Linear Independence and Linear Maps











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1 Linear Independence

The concept of linear independence of a set of vectors in $\mathbb{R}^n$ is extremely important in linear algebra and its applications. A big reason linear dependence is important is because if two (or more) vectors are dependent, then one of them is unnecessary, since the span of the two vectors would be the same as the span of one of the two vectors on their own. What does this really mean? If the vectors are linearly dependent, then one or more of them is just duplicating information, and there's a more mathematically elegant way to express the same set of vectors (see section 1.1 on Span for more details). Formally we write this as $a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2} + a_3\boldsymbol{v_3} + a_4\boldsymbol{v_4} = \boldsymbol{0} \Leftrightarrow a_1 = a_2 = a_3 = a_4 = 0$ hence a trivial solution.

In the theory of vector spaces, a set of vectors is said to be linearly dependent if any one vector can be expressed as some combination of the others. In other words, there is a nontrivial linear combination of the vectors that equals the zero vector. Non-trivial means that $a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2} + a_3\boldsymbol{v_3} + a_4\boldsymbol{v_4} = \boldsymbol{0} \nRightarrow a_1 = a_2 = a_3 = a_4 = 0$. If no such linear combination exists, then the vectors are said to be linearly independent. These concepts are central to the definition of dimension.

Way 1: using RREF	**Way 2: Using the determinant**
Step 1: We set equal to zero $a_1\boldsymbol{v_1}+a_2\boldsymbol{v_2} + a_3\boldsymbol{v_3} + a_4\boldsymbol{v_4} = \boldsymbol{0}$ Step 2: Do RREF on the augmented matrix Step 3: If we end up with the **identity matrix** (if square matrix that is) then this implies all coefficients must be zero $a_1 = a_2 = a_3 = a_4 = 0$ then **linearly independent** If we get **all zeros on the bottom** row of a square matrix then **linearly dependent** Watch out though: e.g. we end up with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. This is not a square matrix and has zero rows, but it is linearly independent, NOT dependent since solving gives $a1 = 0$ and $a_2 = 0$. **So, if the number of leading ones in RREF is equal to the number of variables/columns' instead, then linearly independent.** So In summary: Linearly independent: $a_1v_1+a_2v_2 + a_3v_3 + a_4v_4 = 0 \Leftrightarrow a_1 = a_2 = a_3 = a_4 = 0$ i.e. linearly independent since get a trivial solution Linearly dependent: $a_1v_1+a_2v_2 + a_3v_3 + a_4v_4 = 0$ gives us a non-trivial solution i.e not all coefficients are zero	We can only use this method if the matrix is square. It also can take a long time for a large matrix Form Matrix based on the vectors Linearly Independent: determinant $\neq 0$ Linearly Dependent: determinant =0 Note: This is equivalent to checking whether the matrix is invertible since if invertible the det $\neq 0$ and hence linearly independent

$\underline{v_1} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \underline{v_2} = \begin{pmatrix}1\\1\\1\\5\end{pmatrix}, \underline{v_3} = \begin{pmatrix}1\\1\\3\\5\end{pmatrix}, \underline{v_4} = \begin{pmatrix}1\\2\\3\\5\end{pmatrix}$

Decide whether $\{v_1, v_2, v_3, v_4\}$are linearly dependent or dependent over $\mathbb{Q}$

Note: Remember that $\mathbb{Q}$ is the set of rational numbers.

Way 1:

Consider $a_1\boldsymbol{v_1}+a_2\boldsymbol{v_2} + a_3\boldsymbol{v_3} + a_4\boldsymbol{v_4} = \boldsymbol{0}$

$$\begin{pmatrix}1&1&1&1\\1&1&1&2\\1&1&3&3\\1&5&5&5\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Form augmented matrix

$$\left(\begin{array}{cccc|c}1&1&1&1&0\\1&-1&1&2&0\\1&1&3&3&0\\1&5&5&5&0\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&1&1&1&0\\0&0&0&1&0\\0&0&2&2&0\\0&4&4&4&0\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&1&1&1&0\\0&0&0&1&0\\0&0&1&1&0\\0&1&1&1&0\end{array}\right) \Rightarrow$$

$$\left(\begin{array}{cccc|c}1&1&1&1&0\\0&1&1&1&0\\0&0&1&1&0\\0&0&0&1&0\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&1&0\\0&0&1&1&0\\0&0&0&1&0\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\end{array}\right) \Rightarrow$$

$$\left(\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right)$$

Note: operations done were

E(2,1;-1) and E(3,1;-1) and E(4,1;-1), D(3,$\frac{1}{2}$) and D(4 ,$\frac{1}{4}$), P(2,4), E(1,2;−1), E(2,3;−1), E(3,4;-1)

We know this is linearly independent since we got the identity matrix

But let's write it properly,

$a_1 = 0$

$a_2 = 0$

$a_3 = 0$

$a_4 = 0$

$a_1v_1+a_2v_2 + a_3v_3 + a_4v_4 = 0 \Longrightarrow a_1 = a_2 = a_3 = a_4 = 0$

i.e. the only way we end up with the zero vector is with trivial coefficients

Therefore linearly independent

Way 2: Determinant

$$\left|\begin{pmatrix}1&1&1&1\\1&1&1&2\\1&1&3&3\\1&5&5&5\end{pmatrix}\right|$$

Using a calculator we get

$det = -8 \neq 0$

∴ linearly independent

Note: You're commonly also asked these types questions over $\mathbb{F}_n$ not just $\mathbb{Q}$. This is easy. Just edit the numbers in the original vector by first and proceed as above (see the chapter on fields)

$\underline{v_1} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \underline{v_2} = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \underline{v_3} = \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \underline{v_4} = \begin{pmatrix}1\\-1\\3\\1\end{pmatrix}$. Decide whether $\{v_1, v_2, v_3, v_4\}$ are linearly dependent or dependent over $\mathbb{Q}$. If linearly dependent, find the dependence relation between them.

Way 1:

If LI: $a_1v_1+a_2v_2 + a_3v_3 + a_4v_4 = \underline{0} \Longrightarrow a_1 = a_2 = a_3 = a_4 = 0$

$$\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&3\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\left(\begin{array}{cccc|c}1&1&1&1&0\\1&-1&1&-1&0\\1&1&-1&3&0\\1&-1&-1&1&0\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&1&1&1&0\\0&-2&0&-2&0\\0&0&-2&2&0\\0&-2&-2&0&0\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&1&1&1&0\\0&1&0&1&0\\0&0&-2&2&0\\0&1&1&0&0\end{array}\right) \Rightarrow$$

$$\left(\begin{array}{cccc|c}1&0&1&0&0\\0&1&0&1&0\\0&0&-2&2&0\\0&0&1&-1&0\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&0&1&0&0\\0&1&0&1&0\\0&0&1&-1&0\\0&0&1&-1&0\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&0&0&1&0\\0&1&0&1&0\\0&0&1&-1&0\\0&0&0&0&0\end{array}\right)$$

Note: operations done were

E(2,1;-1) and E(3,1;-1) and E(4,1;-1), D(2,$-\frac{1}{2}$) and D(4,$-\frac{1}{2}$) , E(1,2;−1) and E(4,2;−1)

D(3,$-\frac{1}{2}$), E(1,3;−1) and E(4,3;−1)

Way 2: Determinant

$$\left|\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&3\\1&-1&-1&1\end{pmatrix}\right|$$

Using a calculator we get

$det = 0$

∴ linearly dependent

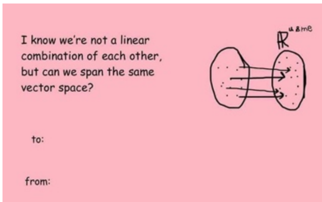
We knew this is linearly dependent since got the last row being all zeros.

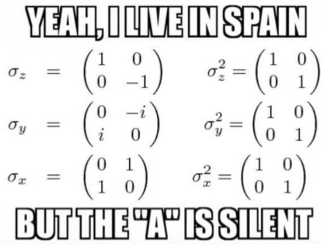
But let's write it properly to find the actual dependence relation,
$a_1 + a_4 = 0$
$a_2 + a_4 = 0$
$a_3 - a_4 = 0$

Re-arrange for leading 1 variables like we did when finding a general solution
$a_1 = -a_4$
$a_2 = -a_4$
$a_3 = a_4$
$a_4 = a_4$
Pick any values for the non-leading variables
Here we only have one uncircled variable which is a_4:
Choose $a_4 = 1$: so $a_1 = -1, a_2 = -1, a_3 = 1$
$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$ gives us $-v_1 - v_2 + a_3v_3 + a_4v_4 = 0$

So found a NON-trivial way to get $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ hence dependent.

1.1 Basis and Spanning Set





Linear algebra is all about a basis!

Consider this:
We can describe all of the coordinate plane using only (a,0)+(0,b). We just need two variables, that's why it's called 2-dimensional. One variable, like (a,a) wouldn't be enough since we can't make (2,3) with that setup, and having more than two is overkill. As long as the two components are fundamentally different, we can even use weird combos, like (a,0)+(b,b) to create any coordinate pair.

A **spanning set** for a space is a set of vectors from which you **can make every vector in the space** by **using addition and scalar multiplication** (i.e. by taking "linear combinations"). ... A **basis** for a space is a spanning set with the extra property that the **vectors are linearly independent**, so a basis means linearly independent and simply put are the "vectors/elements you can use to describe things". Given a basis of a vector space, every element of the vector space can be expressed uniquely as a finite linear combination of basis vectors, whose coefficients are called vector coefficients.

The basis of a Vector Space is the smallest set of vectors such that they span the entire Vector Space. The canonical basis for $\mathbb{R}^3$ *is* $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. You will see most textbooks and lecturers denote this with the letter e or $\in$. This looks like. $\in_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\in_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\in_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$. This means specifically that the vectors are all zero, except for their index value, which is 1. For example, $\in_5 = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$. These vectors only change based on how many dimensions they're in, meaning the number of elements in the vector changes, but the structure of it does not, much like how an Identity matrix can be a square matrix of any size with the same structure even as the number of elements changed.

You can create any vector in $\mathbb{R}^3$ by a linear combination of those three vectors i.e $\begin{pmatrix}a\\b\\c\end{pmatrix}$ can be written as a linear combination of $a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}$. This standard basis is obvious. Are there any other vectors that form a basis of $\mathbb{R}^3$? Yes! There is more than one possible basis for $\mathbb{R}^3$. This is not unique basis. Another basis for the same space is $\left\{\begin{pmatrix}-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. A spanning set for the same space is $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\\frac{1}{2}\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$, but this set is **not a basis** this time, because it is linearly dependent. We know this without even checking because we only need 3 vectors for it to be a basis in $\mathbb{R}^3$ and here we have 4. In other words, the basis is the **minimum** spanning set. For example, take another spanning set. $\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$. One of the vectors is a linear combination of the others $\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix}$ so we can drop it from the set and still have the same span, so this first spanning set would not be a basis, but the spanning set $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ would be a basis. Let's now take a look at a set which is not a spanning set. The set $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ is not a spanning set since we could only obtain a last component of zero i.e. vectors of the form $\begin{pmatrix}x\\y\\0\end{pmatrix}$. In other words, no linear combination of multiples of the given vectors will ever contribute anything other than 0 in the z dimension, and therefore all vectors spanned have $z = 0$. One last example to make sure you understand. Consider the vector $\begin{pmatrix}4\\2\\1\end{pmatrix}$. How can we write it using the basis $\left\{\begin{pmatrix}3\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$?

$\begin{pmatrix}4\\2\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}3\\0\\0\end{pmatrix} + 3\begin{pmatrix}1\\1\\0\end{pmatrix} - 1\begin{pmatrix}0\\1\\-1\end{pmatrix}$. Here we just showed for one particular element, but you can see that from the basis you can get any element!

The dimension of a vector space is the number of elements in the basis of the vector space. For example, in $\mathbb{R}^3$ we need 3 vectors to be a basis. If we have more vectors it can't be a basis. Dimensions and ranks will be discussed in a bit more detail in sections 1.2 onwards.

Common questions that come up are:

- Proving whether something is a basis/ showing whether it is a basis
- Finding a basis

$v_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix}, v_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, v_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$

Show that $\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ following are basis for $\mathbb{F}^3$

We need to show spans and LI for a basis

- **To show LI:**

  Way 1: Check if get the identity matrix

  If LI: $a_1v_1 + a_2v_2 + a_3v_3 = \underline{0} \Rightarrow a_1 = a_2 = a_3 = a_4 = 0$

  $$\begin{pmatrix}1&0&1\\-1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

  $$\left(\begin{array}{ccc|c}1&0&1&0\\-1&1&0&0\\-1&1&1&0\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&1&0&0\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&-1&0\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&1&0\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)$$

  $\Rightarrow a_1 = a_2 = a_3 = 0 \therefore$ LI

  Way 2: Use determinant

  $$det = \left|\begin{pmatrix}1&0&1\\-1&1&0\\1&1&1\end{pmatrix}\right| = -1 \neq 0 \therefore \text{LI}$$

- **To show spans**

  If spans: $a_1v_1 + a_2v_2 + a_3v_3 = \underline{x}$ i.e $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = a_1\begin{pmatrix}1\\-1\\-1\end{pmatrix} + a_2\begin{pmatrix}0\\1\\1\end{pmatrix} + a_3\begin{pmatrix}1\\0\\1\end{pmatrix}$

  Want to get coefficients a_1, a_2, a_3 in terms of x_1, x_2 and x_3

  Way 1:

  $x_1 = a_1 + a_3 \Rightarrow a_1 = x_1 - a_3$
  $x_2 = -a_1 + a_2 \Rightarrow a_2 = x_2 + a_1 = x_2 + x_1 - a_3$
  $x_3 = -a_1 + a_2 + a_3 \Rightarrow a_3 = x_3 - a_1 - a_2 = x_3 - (x_1 - a_3) - (x_2 + x_1 - a_3) = x_3 - x_1 + a_3 - x_2 - x_1 + a_3$
  $= x_3 - 2x_1 - x_2 + 2a_3$

  $a_3 = 2x_1 + x_2 - x_3$
  $a_2 = x_2 + x_1 - 2x_1 - x_2 + x_3 = -x_1 + x_3$
  $a_1 = x_1 - 2x_1 - x_2 + x_3 = -x_1 - x_2 + x_3$

  $(-x_1 - x_2 + x_3)\boldsymbol{v_1} + (-x_1 + x_3)\boldsymbol{v_2} + (2x_1 + x_2 - x_3)\boldsymbol{v_3} = \boldsymbol{x} \therefore$ spans

  Note: could have use elimination:
  $x_1 = a_1 + a_3$
  $x_2 = -a_1 + a_2$
  $\underline{x_3 = -a_1 + a_2 + a_3}$

  Way 2: Form augmented matrix and get into RREF

  $$\left(\begin{array}{ccc|c}1&0&1&x_1\\-1&1&0&x_2\\-1&1&1&x_3\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&0&1&x_1\\0&1&1&x_1+x_2\\0&1&0&x_3-x_1\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&0&1&x_1\\0&1&1&x_1+x_2\\0&0&-1&-2x_1-x_2+x_3\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&0&1&x_1\\0&1&1&x_1+x_2\\0&0&1&2x_1+x_2-x_3\end{array}\right) \Rightarrow$$

  $$\left(\begin{array}{ccc|c}1&0&0&-x_1-x_2+x_3\\0&1&0&-x_1+x_3\\0&0&1&2x_1+x_2-x_3\end{array}\right)$$

  $a_1 = -x_1 - x_2 + x_3$
  $a_2 = -x_1 + x_3$
  $a_3 = 2x_1 + x_2 - x_3$
  $(-x_1 - x_2 + x_3)\boldsymbol{v_1} + (-x_1 + x_3)\boldsymbol{v_2} + (2x_1 + x_2 - x_3)\boldsymbol{v_3} = \boldsymbol{x} \therefore$ span

$\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ is a basis since LI and spans

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$v_1=\begin{pmatrix}0\\1\\-2\end{pmatrix}, v_2=\begin{pmatrix}5\\-7\\4\end{pmatrix}, v_3=\begin{pmatrix}6\\3\\5\end{pmatrix}$. Is $\{v_1,v_2,v_3\}$ a basis for $\mathbb{R}^3$?

We need to show spans and LI for a basis

- **To show LI:**

  Way 1: Check if get the identity matrix

  If LI: $a_1v_1+a_2v_2+a_3v_3=\underline{0} \Longrightarrow a_1=a_2=a_3=a_4=0$

  $$\begin{pmatrix}0&5&6\\1&-7&3\\-2&4&5\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

  $$\begin{pmatrix}0&5&6&|0\\1&-7&3&|0\\-2&4&5&|0\end{pmatrix}\Longrightarrow\begin{pmatrix}1&-7&3&|0\\0&5&6&|0\\-2&4&5&|0\end{pmatrix}\Longrightarrow\begin{pmatrix}1&-7&3&|0\\0&5&6&|0\\0&-10&11&|0\end{pmatrix}\Longrightarrow\begin{pmatrix}1&-7&3&|0\\0&5&6&|0\\0&0&23&|0\end{pmatrix}\Longrightarrow\begin{pmatrix}1&-7&3&|0\\0&1&\frac{6}{5}&|0\\0&0&1&|0\end{pmatrix}$$

  $$\Longrightarrow\begin{pmatrix}1&0&0&|0\\0&1&0&|0\\0&0&1&|0\end{pmatrix}\Longrightarrow a_1=a_2=a_3=0$$

  Way 2: Use determinant

  $$det=\left|\begin{pmatrix}0&5&6\\1&-7&3\\-2&4&5\end{pmatrix}\right|=-115\neq 0$$

- **To show spans**

  If spans: $a_1v_1+a_2v_2+a_3v_3=\underline{x}$ i.e $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=a_1\begin{pmatrix}0\\1\\-2\end{pmatrix}+a_2\begin{pmatrix}5\\-7\\4\end{pmatrix}+a_3\begin{pmatrix}6\\3\\5\end{pmatrix}$

  Way 1:

  $x_1=5a_2+6a_3$
  $x_2=a_1-7a_2+3a_3$
  $x_3=-2a_1+4a_2+5a_3$

  Want to get coefficients a_1,a_2,a_3 in terms of x_1,x_2 and x_3

  We need to find the $a's$ in terms of the $x's$ only. This is a bit annoying to do since 2 equations have 3 unknowns in them and most examples are easier than this, so lets do the way below instead

  Way 2: Form augmented matrix and get into RREF

  $$\begin{pmatrix}0&5&6&|\ x_1\\1&-7&3&|\ x_2\\-2&4&5&|\ x_3\end{pmatrix}$$

  Now let's get into RREF

  $$\begin{pmatrix}1&-7&3&|\ x_2\\0&5&6&|\ x_1\\-2&4&5&|\ x_3\end{pmatrix}\Longrightarrow\begin{pmatrix}1&-7&3&|\ x_2\\0&5&6&|\ x_1\\0&-10&11&|\ 2x_2+x_3\end{pmatrix}\Longrightarrow\begin{pmatrix}1&-7&3&|\ x_2\\0&5&6&|\ x_1\\0&0&23&|\ 2x_1+2x_2+x_3\end{pmatrix}\Longrightarrow$$

  $$\begin{pmatrix}1&-7&3&|\ x_2\\0&1&\frac{6}{5}&|\ \frac{x_1}{5}\\0&0&1&|\ \frac{2x_1+2x_2+x_3}{23}\end{pmatrix}$$

  $$\Longrightarrow\begin{pmatrix}1&0&\frac{57}{5}&|\ \frac{7}{5}x_1+x_2\\0&1&\frac{6}{5}&|\ \frac{x_1}{5}\\0&0&1&|\ \frac{2x_1+2x_2+x_3}{23}\end{pmatrix}\Longrightarrow\begin{pmatrix}1&0&\frac{57}{5}&|\ \frac{7}{5}x_1+x_2\\0&1&0&|-\frac{6}{5}\left(\frac{2x_1+2x_2+x_3}{23}\right)+\frac{x_1}{5}\\0&0&1&|\ \frac{2x_1+2.x_2+x_3}{23}\end{pmatrix}\Longrightarrow\begin{pmatrix}1&0&0&|-\frac{57}{5}\left(\frac{2x_1+2x_2+x_3}{23}\right)+\frac{7}{5}x_1+x_2\\0&1&0&|-\frac{6}{5}\left(\frac{2x_1+2x_2+x_3}{23}\right)+\frac{x_1}{5}\\0&0&1&|\ \frac{2x_1+2x_2+x_3}{23}\end{pmatrix}$$

  $a_1=-\frac{57}{5}\left(\frac{2x_1+2x_2+x_3}{23}\right)+\frac{7}{5}x_1+x_2$
  $a_2=-\frac{6}{5}\left(\frac{2x_1+2x_2+x_3}{23}\right)+\frac{x_1}{5}$
  $a_3=\frac{2x_1+2x_2+x_3}{23}$

  $$\left(-\frac{57}{5}\left(\frac{2x_1+2x_2+x_3}{23}\right)+\frac{7}{5}x_1+x_2\right)\boldsymbol{v_1}+\left(-\frac{6}{5}\left(\frac{2x_1+2x_2+x_3}{23}\right)+\frac{x_1}{5}\right)\boldsymbol{v_2}+\left(\frac{2x_1+2x_2+x_3}{23}\right)\boldsymbol{v_3}=\boldsymbol{x}$$

Find a basis for the span of the vectors $v_1=\begin{pmatrix}1\\2\\1\end{pmatrix}, v_2=\begin{pmatrix}-1\\-1\\1\end{pmatrix}, v_3=\begin{pmatrix}1\\4\\5\end{pmatrix}, v_4=\begin{pmatrix}3\\4\\-1\end{pmatrix}, v_5=\begin{pmatrix}0\\1\\2\end{pmatrix}, v_6=\begin{pmatrix}4\\9\\6\end{pmatrix}$

We can use the fact that if we put into RREF, look where the columns of the leading ones are and the corresponding columns from the original matrix form the basis

So we are trying to find a basis for $\left\{\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix},\begin{pmatrix}1\\4\\5\end{pmatrix},\begin{pmatrix}3\\4\\-1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}4\\9\\6\end{pmatrix}\right\}$

Form corresponding augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 & 3 & 0 & 4 & |0 \\ 2 & -1 & 4 & 4 & 1 & 9 & |0 \\ 1 & 1 & 5 & -1 & 2 & 6 & |0 \end{pmatrix}$$

Doing all the necessary row operations to give us RREF produces:

$$\begin{pmatrix} 1 & 0 & 3 & 1 & 1 & 5 & |0 \\ 0 & 1 & 2 & -2 & 1 & 1 & |0 \\ 0 & 0 & 0 & 0 & 0 & 0 & |0 \end{pmatrix}$$

Look for corresponding columns from original matrix from where columns here that have leading ones

These are $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

So $\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$ spans $\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 6 \end{pmatrix}\right\}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ are LI

so $\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$ is a basis for $\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 6 \end{pmatrix}\right\}$

Might be more bases out there, most likely not the only one as not unique! There is no such thing as a unique basis. Any vector can be replaced with some scalar multiple of itself, even leaving aside rotations and linear combinations.

Note: If not convinced whether $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ are LI

RREF of $\begin{pmatrix} 1 & -1 & |0 \\ 2 & -1 & |0 \\ 1 & 0 & |0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & |0 \\ 0 & 1 & |0 \\ 0 & 0 & |0 \end{pmatrix}$. It's not the identity matrix since its not square, but we get both coefficients being zero

Show that the following are basis for $\mathbb{F}^3$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Ans. LI and $(-x_1 - x_2 + x_3)\boldsymbol{v_1} + (-x_1 + x_3)\boldsymbol{v_2} + (2x_1 + x_2 - x_3)\boldsymbol{v_3} = \boldsymbol{x}$

What is the span of $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right\}$

Hint: Just find the basis

1.2 Further Sub-Spaces Of A Matrix

As previously seen, a basis of a subspace is a set of vectors which can be used to represent any other vector in the subspace.
The set must:

- Be linearly independent i.e. not include any vectors which are linearly dependent upon other vectors in the set.
- Span all of the subspace i.e. so that each vector in the subspace can be uniquely represented as a linear combination of the basis vectors.

1.2.1 Dimension

As also already mentioned, **the dimension of a vector spaces is the number of elements in the basis of the vector space (it is known as the size of the rank)**. So, the dimension of a vector space V is the cardinality (i.e. the number of vectors) of a basis of V over its base field. i.e. number of elements in the basis hence the column space.

1.2.2 Row Space

Consider m × n matrix A. If A is an m × n matrix with real entries, the row space of A is the subspace of $\mathbb{R}^n$ spanned by its rows.
It is important to realise that elementary row operations do not change the row space and in general, the rows of a matrix may not be linearly independent.
The nonzero rows of any row-echelon form of A is a basis for its row space. So, we just row reduce, identify non zero rows and these form the basis for the row space.

1.2.3 Column Space

This is the number of elements in basis. If A is an m × n matrix with real entries, the column space of A is the subspace of $\mathbb{R}^m$ spanned by its columns. If A is an m × n matrix with real entries, the set of column vectors of A corresponding to those columns containing leading ones in any row-echelon form of A is a basis for the column space of A. Obviously, the column space of A equals the row space of A^T , so a basis can be computed by reducing A^T to row-echelon form. However, this is not the best way. We don't need to go all the way to RREF this time; we can see where the leading ones will be just from REF. So, we just identify pivot columns in the RREF and find the corresponding columns from the original matrix.

1.2.4 Rank

The rank of a matrix is the maximum number of its linearly independent column vectors (or row vectors). The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its row echelon matrix. Therefore, to **find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of non-zero rows**. Sometimes the rank of a matrix is this is more confusing explained as the dimension of the vector space generated (or spanned) by its columns. In other words, the image of the linear transformation represented by the matrix. The image is the column space of the matrix, so the rank is the dimension of the column space, and consequently equal to the number of linearly independent columns.

Consider the matrix A $\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$ where $T: \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ defined by $\mathrm{T}(x) = Ax$

Note: This just means Let's define T as the transformation which is induced by **LEFT** multiplication by A

In RREF this is

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Row space basis $\{(1 \quad -2 \quad 0 \quad -1 \quad 3), (0 \quad 0 \quad 1 \quad 2 \quad -2)\}$

Column space basis: $\left\{\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}\right\}$

Row rank: 2
Column rank: 2
Rank of matrix A = 2 since 2 non-zero rows

1.2.5 Kernel (Nullspace) and Image

Consider an $\mathrm{m} \times \mathrm{n}$ matrix A.

The kernel or nullspace of a matrix A, denoted ker(A), is the set of all vectors $\boldsymbol{x}$ for which A$\boldsymbol{x}$=**0**. You can think of this verbally as the set of all inputs which are taken to zero or the vectors that 'go in' that give zero or the set of all points in $\mathbb{R}^{\mathrm{m}}$ such that multiplying this matrix A with them gives the zero vector. We write this formally as,

$$\mathrm{Ker}\ (\mathrm{T}) = \{\mathrm{x} \in \mathbb{R}^{\mathrm{m}} | \mathrm{A}\boldsymbol{x} = \mathbf{0}\}$$

This is also sometimes written as let $T: V \rightarrow W$ be a linear transformation

$$\mathrm{Ker}\ (\mathrm{T}) = \{\mathbf{v} \in \mathrm{V}: \mathrm{T}(\mathbf{v}) = \mathbf{0}\ \}$$

The nullity of a matrix is the dimension of its nullspace. This is the dimension of the kernel.

The image is the set of all outputs of the linear mapping i.e. the set of all points in $\mathbb{R}^{\mathrm{n}}$ that you get by multiplying this matrix to points in $\mathbb{R}^{\mathrm{m}}$. You can find these by checking the matrix on the standard basis. When you play around with the mapping a little bit then you should find that the image is in fact a very familiar subspace associated with the matrix. The image is the same as the basis!
We write this formally as,

$$\mathrm{Im}\ (\mathrm{T}) = \{\boldsymbol{x} \in \mathbb{R}^{\mathrm{n}} \text{ such that } \mathrm{A}\boldsymbol{x} \in \mathbb{R}^{\mathrm{m}}\}$$

This is also sometimes written as let $T: V \rightarrow W$ be a linear transformation

$$\mathrm{Im}\ (\mathrm{T}) = \{\mathbf{w} \in \mathrm{W}: \text{for some } \mathbf{v} \in \mathrm{V}, \mathrm{T}(\mathbf{v}) = \mathbf{w}\ \}$$

The set Im(T) consists of all elements in W which are hit by the mapping T. In other words "the set of all vectors that are hit by the mapping T".

It is important know that:

- Dim(Ker)+Dim(Image)=Dim(Vector space)
  So, this means that Dim(Ker) and Dim(Image) are given by basis (how many elements in the basis)
- Theorem (Rank-Nullity Theorem): For any $m \times n$ matrix A, rank(A) + nullity(A) = number of columns (n)
- Row rank=column rank= matrix rank

This will all be made clear with an example.

Let T: $\mathbb{Q}^7 \rightarrow \mathbb{Q}^4$ be the linear Map given by $T(\underline{x}) = A\underline{x}$ where A=$\begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 & 0 \\ 1 & 1 & -1 & 1 & 3 & 4 & 6 \\ 1 & -1 & 1 & 1 & -1 & -2 & -4 \\ 1 & -1 & -1 & -1 & 1 & 0 & 2 \end{pmatrix}$

Find a basis for $Ker(T)$ and a basis for $Im(T)$

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 & 0 \\ 1 & 1 & -1 & 1 & 3 & 4 & 6 \\ 1 & -1 & 1 & 1 & -1 & -2 & -4 \\ 1 & -1 & -1 & -1 & 1 & 0 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 & 0 \\ 0 & 0 & -2 & -2 & 2 & 2 & 6 \\ 0 & -2 & 0 & -2 & -2 & -4 & -4 \\ 0 & -2 & -2 & -4 & 0 & -2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 0 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & -1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 & 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & -1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 & -1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We set equal to zero since the kernel are the vectors that go in to give 0

$x_1 + x_4 + x_5 + x_6 + x_7 = 0$

$x_2 + x_4 + x_5 + 2x_6 + 2x_7 = 0$
$x_3 + x_4 - x_5 - x_6 - 3x_7 = 0$

So, we write the leading 1 fixed variables (x_1, x_2, x_3) in terms of the free variables (x_4, x_5, x_6, x_7) as usual like when solving a system generally
See section **Error! Reference source not found.** if you can't remember how to solve generally.

$x_1 = -x_4 - x_5 - x_6 - x_7$
$x_2 = -x_4 - x_5 - 2x_6 - 2x_7$
$x_3 = -x_4 + x_5 + x_6 + 3x_7$
$x_4 = x_4$
$x_5 = x_5$
$x_6 = x_6$
$x_7 = x_7$

The number of free variables tell us the dim which is 4

General solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = x_4 \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} -1 \\ -2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

So, we have written the kernel as a span of 4 linearly independent vectors

Basis Kernel/Nullspace= $\left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Nullity =Dim Ker (T)=number of free variables=number of basis elements= 4

To find the Im(Ker) we look for which columns the leading ones occur in the row reduced matrix. They occur in columns 1, 2 3. We then find the corresponding columns of the matrix A

Basis for Im(T)= $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1\ - \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$

Dim Im(T)= 3
Dimension of vector space = 7
4+3=7. This checks out!

When A is an $m \times n$ matrix over a field $\mathbb{F}$, we define $K_A = \left\{ x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \in \mathbb{F}^n : A\boldsymbol{x} = 0 \right\}$

Find $\dim_{\mathbb{F}} K_A$ and a basis for K_A over $\mathbb{F}$. Find the dimension of the Image and a basis for the image.

$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$ $\mathbb{F} = \mathbb{F}_3$

In $\mathbb{F}_3 = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
This is already in RREF

We set equal to zero since the kernel are the vectors that go in to give 0
$x_1 + 2x_2 + x_3 + 2x_4 = 0$

So, we write the leading 1 fixed variable (x_1) in terms of the free variables (x_2, x_3, x_4) as usual like when solving a system generally
See section 5.5 if you can't remember how to solve generally.

$x_1 = -2x_2 - x_3 - 2x_4$
$x_2 = x_2$
$x_3 = x_3$
$x_4 = x_4$

The number of free variables tell us the dim which is 3

General solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2x_2 - x_3 - 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

So we have written the kernel as a span of 3 linearly independent vectors

Basis Kernel/Nullspace= $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Why do we get that for the basis?

We have $\begin{pmatrix} -2x_2 - x_3 - 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

First choice: $x_2 = 1, x_3 = 0, x_4 = 0 \Longrightarrow \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Second choice: $x_2 = 0, x_3 = 1, x_4 = 0 \Longrightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Third choice: $x_2 = 0, x_3 = 0, x_4 = 1 \Longrightarrow \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Hence we get the result we got for the basis

Nullity =Dim Ker (T)= number of free variables = number of basis elements = 3

To find the Im(Ker) we look for which columns the leading ones occur in the row reduced matrix. This only occurs in columns 1. We then find the corresponding columns of the matrix A

Basis for Im(T)= $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

Dim Im(T)= 1

Dimension of vector space $= 3 + 1 = 4$

To try:

Let A be an $m \times n$ matrix over a field $\mathbb{F}$. We define $K_A = \{x = \begin{pmatrix} x_1 \\ . \\ . \\ .x_n \end{pmatrix} \in \mathbb{F}^n, Ax = 0\}$. In each case below find $dim_{\mathbb{F}}(K_A)$ and a basis for K_A over $\mathbb{F}$.

i. $A = (1 \quad -1 \quad 1 \quad -1)$

ii. $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 2 & 2 & 2 \\ 1 & 2 & 4 & 2 \end{pmatrix}$

So, you should now realise that if we have a square $n \times n$ matrix, the following are equivalent:

- A is invertible
- $\det(A) \neq 0$
- The columns of A are linearly independent
- The rows of A are linearly independent
- The columns of A span $\mathbb{R}^n$
- The rows of A span $\mathbb{R}^n$
- The columns of A are a basis in $\mathbb{R}^n$
- The row of A are a basis in $\mathbb{R}^n$
- The reduced row echelon form of A has a leading 1 in each row

So, if you want to check whether any of the conditions are true or false you can pick whichever condition from the list and check it instead.

1.3 Linear Transformations/Maps

A linear transformation is **a function from one vector space to another that respects the underlying (linear) structure of each vector space** (meaning it respects all the axioms of the vector space). A linear transformation is also known as a linear operator or map. The two vector spaces must have the same underlying field.

When studying vector spaces and linear transformations, you can't do anything without a basis. It might sound silly, but when you realize this you will start to understand linear algebra!

Given two vector spaces V and W over a field $\mathbb{F}$ and a function (mapping) $T: V \to W$. We say that T is a linear transformation of V into W (takes V and maps it onto W), if the following 2 properties are true for all $\underline{u} \in U$, $\underline{v} \in V$ and for all scalars $\lambda \in \mathbb{F}$. We say that T is linear if:

- $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$
- $T(\lambda\underline{u}) = \lambda T(\underline{u})$

1.3.1 Finding the Matrix Of Linear Transformations/Maps

Consider the vector $\begin{pmatrix}4\\2\\1\end{pmatrix}$. How can we write it with using the vectors $\begin{pmatrix}3\\0\\0\end{pmatrix}$ *and* $\begin{pmatrix}1\\1\\0\end{pmatrix}$ *and* $\begin{pmatrix}0\\1\\-1\end{pmatrix}$? $\begin{pmatrix}4\\2\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}3\\0\\0\end{pmatrix} + 3\begin{pmatrix}1\\1\\0\end{pmatrix} - 1\begin{pmatrix}0\\1\\-1\end{pmatrix}$.

So, we have expressed the vector $\begin{pmatrix}4\\2\\1\end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix}3\\0\\0\end{pmatrix}$ *and* $\begin{pmatrix}1\\1\\0\end{pmatrix}$ *and* $\begin{pmatrix}0\\1\\-1\end{pmatrix}$. The coefficients were easy to spot, but if you couldn't spot this, you can instead do either of the following:

- write $a\begin{pmatrix}3\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}4\\2\\1\end{pmatrix}$. Form 3 equations $3a + b = 4, b + c = 2$ and $-c = 1$ and use regular algebra to solve for scalars a, b and c.
- form a matrix $\begin{pmatrix}3 & 1 & 0\\0 & 1 & 1\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}4\\2\\1\end{pmatrix}$ and use RREF or inverse multiplication to find scalars a, b and c.

The corresponding vector is $\begin{pmatrix}\frac{1}{3}\\3\\-1\end{pmatrix}$. Here we just showed one particular vector example. We can do this with more vectors to form a matrix!

It is easiest to see this with examples.

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$

$$e_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, e_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\varphi_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \varphi_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \varphi_3 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

What is the matrix representation A for the linear transformation T which expresses e_i vectors in terms of φ_j vectors ?

Formally you might see this question phrased as matrix A as $A = (a_{ji})_{\substack{1\le j\le 3\\1\le i\le 3}}$. Express $\{e_i\}_{1\le i\le 3}$ in terms of $\{\varphi\ \ \}_{1\le j\le 3}$. This means $e_i = \sum_{j=1}^{3} a_{ji}\,\varphi_j$

This is trivial, just put these columns into a vector

$$e_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix} = 3\begin{pmatrix}1\\1\\1\end{pmatrix} - 4\begin{pmatrix}1\\1\\0\end{pmatrix} - 2\begin{pmatrix}-1\\0\\1\end{pmatrix} = 3\varphi_1 - 4\varphi_2 - 2\varphi_3$$

$$e_2 = \begin{pmatrix}0\\1\\1\end{pmatrix} = 0\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\begin{pmatrix}-1\\0\\1\end{pmatrix} = 0\varphi_1 + 1\varphi_2 + 1\varphi_3$$

$$e_3 = \begin{pmatrix}1\\0\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} - 2\begin{pmatrix}1\\1\\0\end{pmatrix} - 1\begin{pmatrix}-1\\0\\1\end{pmatrix} = 2\varphi_1 - 2\varphi_2 - 1\varphi_3$$

$$A = \begin{pmatrix}3 & 0 & 2\\-4 & 1 & -2\\-2 & 1 & -1\end{pmatrix}$$

Check: Apply the matrix T to each of the e_i vectors and see whether you get φ_j vectors

$$\begin{pmatrix} 3 & 0 & 2 \\ -4 & 1 & -2 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 0 & 2 \\ -4 & 1 & -2 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 0 & 2 \\ -4 & 1 & -2 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

You should have realised from this example that applying the transformation T to a vector is the same as multiplying by a matrix A. More formally, for any transformation $T:\mathbb{R}^n \to \mathbb{R}^m$, $T(x) = Ax \Leftrightarrow [\boldsymbol{T}(\boldsymbol{e_1}) \ldots\ldots \boldsymbol{T}(\boldsymbol{e_n})]$ where $\boldsymbol{e_1}, \boldsymbol{e_{,2}}, \ldots \boldsymbol{e_n}$ represent the standard basis i.e. **$\boldsymbol{T}:\mathbb{R}^n \to \mathbb{R}^m$ allows you to form and act upon the vector with an $\boldsymbol{m\ x\ n}$ matrix called A.**

$$T:\mathbb{R}^3 \to \mathbb{R}^4$$

Let $e_i \in \mathbb{R}^3 = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$. Given that: .$T(e_1) = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 0 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} 9 \\ 4 \\ 5 \\ 0 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix}$

What is the matrix representation A for the linear transformation T?

This is trivial, just put these columns into a vector

$$A=\begin{pmatrix} 3 & 9 & 1 \\ 2 & 4 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Check:

$$\begin{pmatrix} 3 & 9 & 1 \\ 2 & 4 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 9 & 1 \\ 2 & 4 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 5 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 9 & 1 \\ 2 & 4 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

$$T:\mathbb{R}^3 \to \mathbb{R}^4$$

Let $b_1 \in \mathbb{R}^3 = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$b_2 \in \mathbb{R}^4 = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

We can work out the transformation matrix T which took elements in $\mathbb{R}^3$ and mapped them onto elements in $\mathbb{R}^4$

Apply to T elements in domain i.e. $\mathbb{R}^3$ to get element in $\mathbb{R}^4$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$
$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

1.3.2 Change of Bases Formula

When using coordinates (and therefore matrices) for linear transformations, it is sometimes helpful to consider a linear transformation with a different basis. Suppose $T: V \to V$ is a linear transformation with matrix N in basis $A = \{a_2, a_2, \dots, a_n\}$ and another basis $B = \{b_1, b, \dots, b_n\}$. Then, if M is the change of basis matrix for A to , the linear transformation has a different matrix in base B namely,

$$M[T]_B^A = M[Id \circ T \circ Id]_B^A = M[Id]_A^A \, M[T]_{B'}^A \, M[Id]_B^{B'}$$

$$\text{Note: } M[T]_B^A = [M[T]_A^B]^{-1}$$

$$T(u) = MNM^{-1}u$$

$M[Id]_B^A$ means the matrix that expresses Id(B's) in terms of A's basis using our previous knowledge of linear transformation matrices.
$M[T]_B^A$ means the matrix that expresses T(B's) in terms of A's basis using our previous knowledge of linear transformation matrices.
$M[T]_A^A$ or $M[Id]_A^A$ sometimes written as $M[T]_A$ or $M[Id]_A$.
Some courses write it as $M[Id]_A^B \uparrow$ or $M[Id]_A^B \downarrow$ to specify which direction it means.
Some write the subscripts on the bottom or left instead of the right.

Let $A = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and $B = \left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ be bases for $\mathbb{Q}^3$

$T: \mathbb{Q}^3 \to \mathbb{Q}^3$ by the linear map $\begin{pmatrix}2x_1 + 2x_2 + x_3\\ -x_2 - x_3\\ x_1 + x_2 + x_3\end{pmatrix}$

Find $M[Id]_B^A$, $M[T]_A^A$, $M[Id]_A^B$, $[MT]_B^B$

Identity map won't change the function: so $ID(b_n) = b_n$

$$ID(b_1) = b_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$ID(b_2) = b_2 = \begin{pmatrix}0\\1\\-1\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$ID(b_2) = b_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$M[Id]_A^B = \begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -1 & 1\end{pmatrix}$$

Notice how this is just the Basis of B as a matrix

$$T(a_1) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2(1)+2(0)+0\\ -0-0\\ 1+0+0\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T(a_1) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2(0)+2(1)+0\\ -1-0\\ 0+1+0\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} - 1\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T(a_1) = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2(0)+2(0)+1\\ -0-1\\ 0+0+1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} - 1\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$M[T]_A^A = \begin{pmatrix}2 & 2 & 1\\ 0 & -1 & -1\\ 1 & 1 & 1\end{pmatrix}$$

Notice how this matrix is just the coefficients of T

We can use a shortcut to get $M[Id]_B^A$ rather than doing in the same way as above

$$M[Id]_B^A = [M[Id]_A^B]^{-1} = \begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$$

We can use the change of base formula to get $M[T]_B^B =$ rather than doing in the same way as above

$$M[T]_B^B = M[Id]_A^B \, M[T]_A^A \, M[Id]_B^A = \begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}2 & 2 & 1\\ 0 & -1 & -1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$$