BioMedical Admissions Test (BMAT)

Section 2: Mathematics

Topic M5: Geometry

Topic M5: Geometry

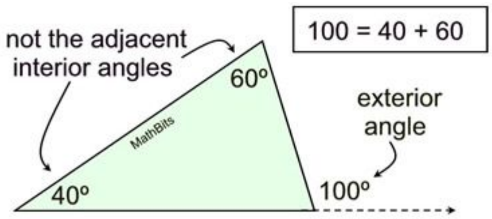
Definitions

- **Points** = a singular position, which can be defined in various ways including by coordinates on a grid or by the intersection point of two lines
- **Lines** = an infinitely long one-dimensional figure
- **Line segments** = an infinitely long one-dimensional figure
- **Vertices** = a corner: for flat shapes, it is where the edges meet and for cones, pyramids etc. all corners (including the point at the top) are called vertices
- **Edges** = the side of a polygon or polyhedron
- **Planes** = flat surfaces
- **Parallel lines** = lines which are always the same perpendicular distance apart - these lines are indicated by arrows
- **Perpendicular lines** = lines that are at right angles to each other - when the lines intersect, the right angle between the lines is generally shown
- **Right angles** = 90°
- **Subtended angles** = an angle subtended by an arc, line segment etc. is one whose two rays pass through the endpoints of the arc, line segment etc
- **Polygons** = closed plane shapes with 3 or more straight sides
- **Regular polygons** = polygons with all its sides equal and all its angles equal
- **Rotational symmetry** = the number of times the shape fits exactly into its outline when turned

Geometry

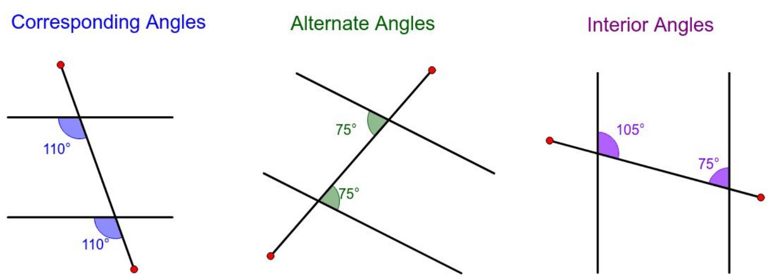
Basic Rules

- Angles in a **triangle** add up to 180°
- Angles on a **straight line** add up to 180°
- Angles in a **quadrilateral** add up to 360°
- Angles **around a point** add up to 360°
- **Exterior** angle of a triangle = sum of opposite interior angles of a triangle
- **Isosceles** triangles have 2 equal sides and hence 2 equal angles



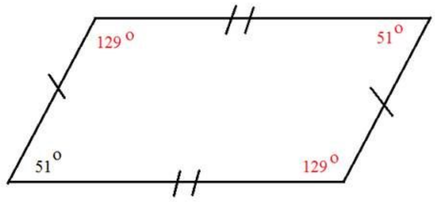
Parallel lines

- **Alternate angles** are equal
- **Corresponding angles** are equal
- **Allied** (also called interior) **angles** add up to 180°



Parallelograms

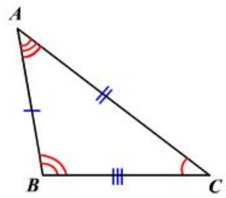
- **Neighbouring** angles add up to 180°
- **Opposite** angles are equal



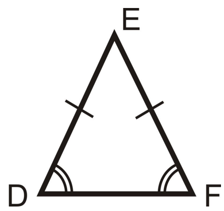
Triangles

Types of triangles

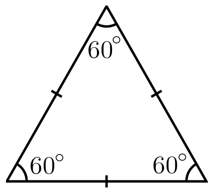
A **scalene triangle** is a triangle where none of the sides (and therefore none of the angles) are equal.



An **isosceles triangle** has two sides which are the same and one that is different. This means that there are two equal angles and one that is different. This can make calculations easy, as we know that the angles in a triangle add up to 180°, so we only need to calculate the one different angle.

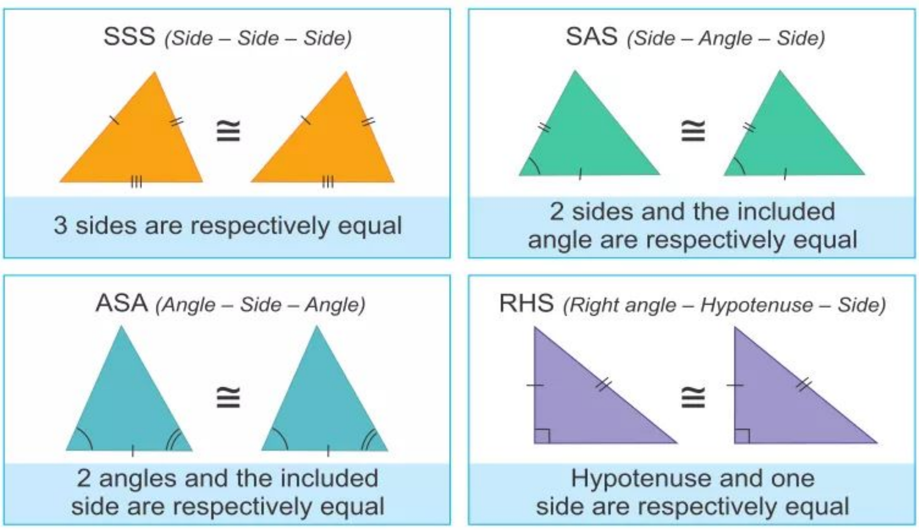


In an **equilateral triangle** all three sides (and therefore all three angles) are the same. As we know that the angles in a triangle add up to 180°, so this means that in an equilateral triangle all angles must add up to 180° ÷ 3 = 60°



Congruence criteria

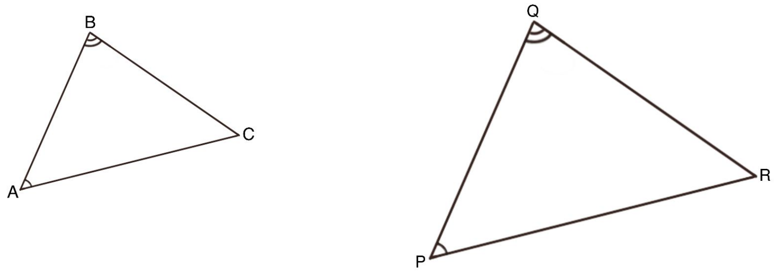
Congruence means that the shapes are the same size and the same shape. There are four ways of determining whether two triangles are congruent:



Similarity

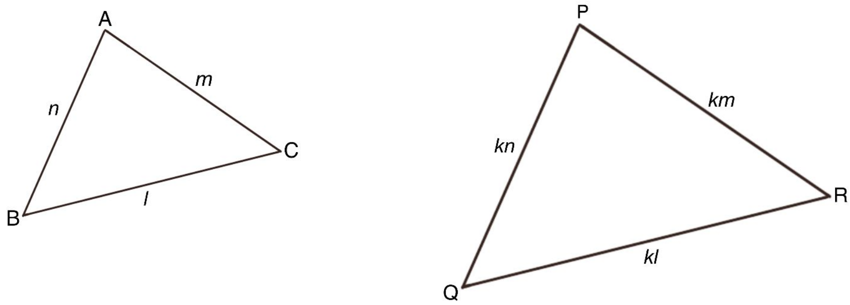
In ∆ABC and ∆PQR, if ∠BAC = ∠QPR, ∠ABC = ∠PQR

Therefore, ∆ABC and ∆PQR are similar



In ∆ABC and ∆PQR, if $\frac{PQ}{AB} = \frac{QR}{BC} = \frac{RP}{CA} = k$, when k is a constant

Therefore, ∆ABC and ∆PQR are similar



Quadrilaterals

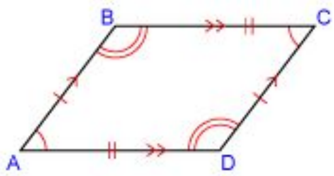
Square

- 2 pairs of parallel sides
- All sides are equal and all interior angles are 90°
- 4 lines of symmetry and order of rotational symmetry is 4

Rectangle

- 2 pairs of parallel sides
- Opposite sides are equal lengths
- All interior angles are 90°
- 2 lines of symmetry and order of rotational symmetry is 2

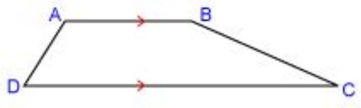
Parallelogram



- 2 pairs of parallel sides
- Opposite sides are equal lengths
- Opposite angles are equal
- Adjacent angles are supplementary
- 0 lines of symmetry and order of rotational symmetry is 2

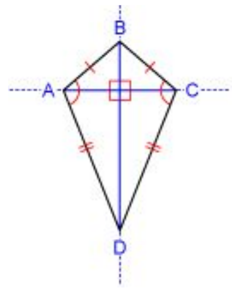
Trapezium

- One pair of parallel sides
- Usually no lines of symmetry



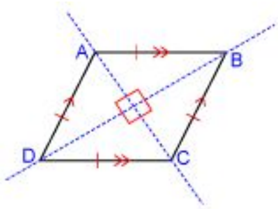
Kite

- 2 pairs of equal side
- One pair of opposite angles which are equal
- 1 line of symmetry
- The diagonals intersect at right angles



Rhombus

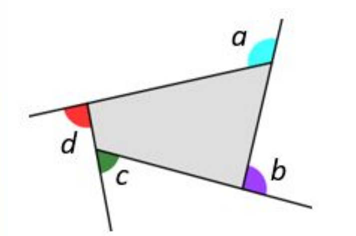
- All four sides are equal
- 2 pairs of parallel sides
- One pair of opposite angles which are equal
- 2 lines of symmetry and rotational symmetry of 2
- The diagonals intersect at right angles



Polygons

External angles

- Sum of exterior angles $= 360°$
- Exterior angle $= \frac{360°}{n}$



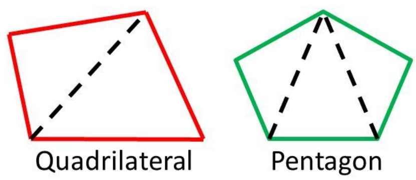
$a + b + c + d = 360°$

Interior angles

- Interior angle $= 180° -$ exterior angle
- Sum of interior angles $= (n - 2) \times 180°$

If you split the polygon into triangles, you can see why we multiply by 180°.

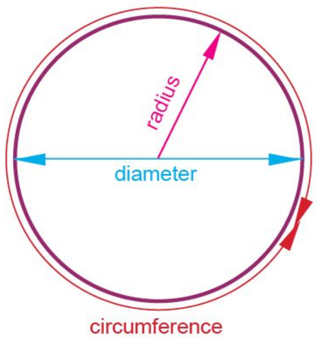
- In a quadrilateral with 4 sides, you can form two triangles, each made up of 180°.
- In a pentagon (5 sides) you can form three triangles.



Circle geometry

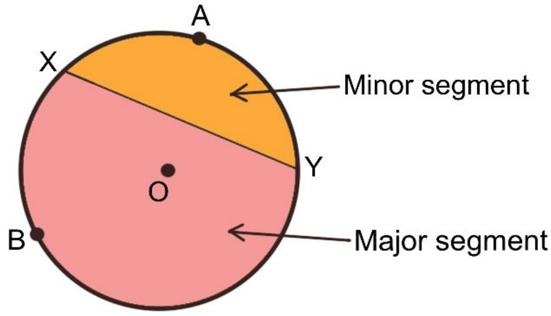
Terms to know

- The **radius** is the length from the centre of the circle to the circumference.
- The **diameter** is the length of a straight line from two points either end of the circumference, which goes through the centre.
- The **circumference** is the distance around the outline of the circle.
- The **tangent** is a line which touches the circle but does not cut through it.



A **chord** is a **line segment** joining two distinct points on the circumference. It divides a circle into two segments.

- The **major segment** refers to the larger segment.
- The **major arc** refers to the circumference of the major segment.
- The **minor segment** refers to the smaller segment.
- The **minor arc** refers to the circumference of the minor segment.

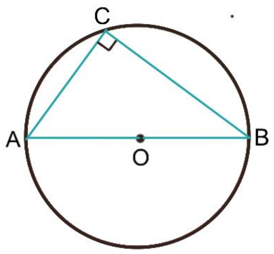


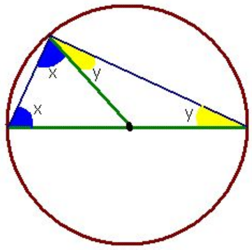
- XY is a chord of the circle
- XAY is the minor arc
- XBY is the major arc
- Region bounded by the chord XY and the minor arc XAY is the minor segment
- Region bounded by the major arc XBY and the chord XY is the major segment

Properties of a circle

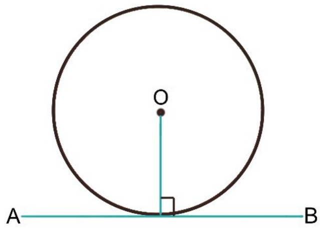
- Angle subtended at the centre is twice the angle subtended at the circumference.

- Angle in a **semicircle** is 90°
  - We can prove that ∠ACB is a right angle by splitting the triangle into two smaller triangles (as on the right)
  - OA=OC=OB as they are all the radius
  - Therefore, we have created two isosceles triangles
  - We can now see that the angle that was at C is now (x+y)
  - We know that angles in a triangle must add up to 180°, so therefore x+(x+y)+y = 180°, which simplifies to 2x+2y = 180°, therefore 2(x+y)=180°
  - This means that x+y = 90°

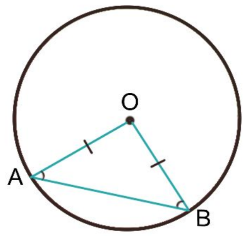




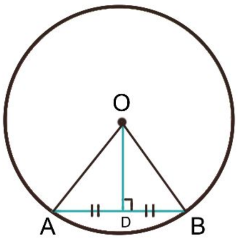
- A **tangent** to a circle is perpendicular to the radius drawn to the point of contact
  - AB is a tangent and X is the point of contact



- 2 radii can be joined to form an **isosceles triangle**
  - ∠ OAB = ∠OBA because they are both base angles of an isosceles triangle



- The **perpendicular bisector of a chord** passes through the centre of a circle
  - As we saw in the last property, the two radii joined to form an isosceles triangle
  - Therefore, a line drawn from centre bisects the chord AB at midpoint D, meaning that AD=AB
  - This means that OD is perpendicular to AB



- Angles in the **same segment** are equal

Transformations

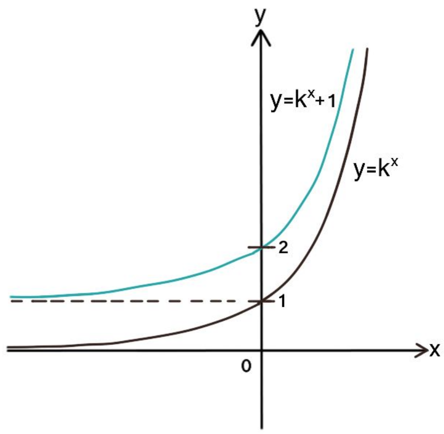
Describing transformations

- The ratio of the areas of two similar figures is equal to the square of the ratio of any two corresponding lengths of the figures.
  - If A_1 and A_2 denote the areas of two similar figures, and l_1 and l_2 denote their corresponding lengths ⇒ $\frac{A_1}{A_2} = (\frac{l_1}{l_2})^2$
- The ratio of the volumes of two similar figures is equal to the cube of the ratio of any two corresponding lengths of the figures.
  - If V_1 and V_2 denote the areas of two similar figures, and l_1 and l_2 denote their corresponding lengths ⇒ $\frac{V_1}{V_2} = (\frac{l_1}{l_2})^3$

Translation

- $y = f(x) + c$ ⇒ Translate by c units along the y-axis

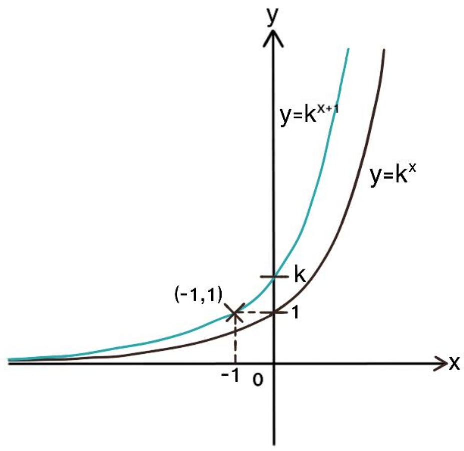
For example, given $y = k^x$, draw $y = k^x + 1$



The entire graph is translated by 1 unit along the y-axis.
The y-intercept shifts up by 1 unit and the asymptote also shifts up from y=0 to y=1.

- $y = f(x + c)$ ⇒ Translate by $-$c units along the x-axis

Example: given $y = k^x$, draw $y = k^{x+1}$

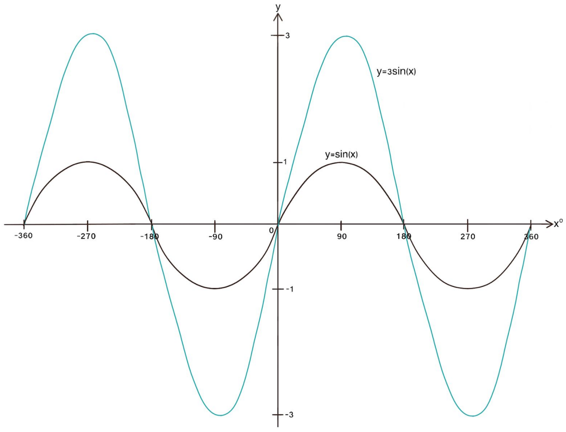


The entire graph is translated by -1 unit along the x-axis.
The y-intercept changes to k: when x=0, y=k^1
And y=1 when $k^{x+1} = 1$ ⇒ let x+1=0,x=-1

Scale

$y = k \times f(x)$ ⇒ Scale by a factor of k along the y-axis

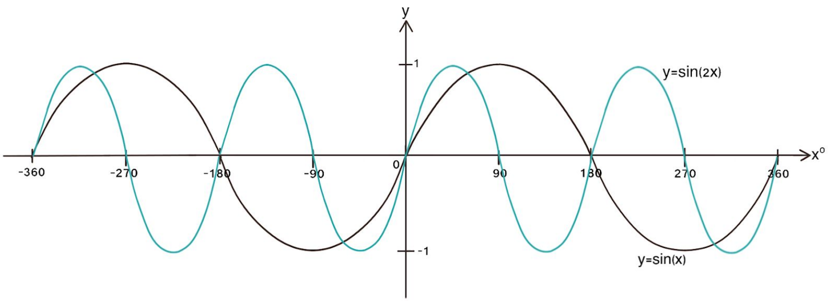
Example: given $y = sin(x)$, draw $y = 3sin\ (x)$.



The entire graph is scaled by a factor of 3 along the y-axis. At each value of x, the value of y is tripled. The range then changes from -1 to 1 ⇒ -3 to 3.

$y = f(kx)$ ⇒ Scale by a factor of $\frac{1}{k}$ along the x-axis.

For example, given $y = sin(x)$, draw $y = sin\ (2x)$.

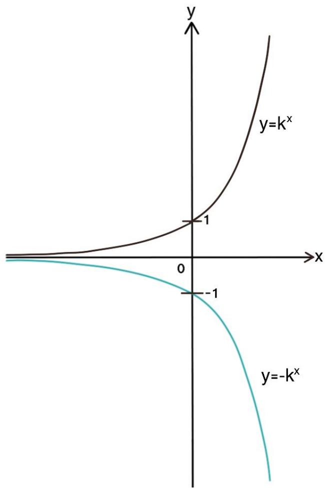


The entire graph is scaled by a factor of $\frac{1}{2}$ along the x-axis. At each value of y, the value of x is halved. A complete sine wave then runs from x values of 0° to 180° instead of 0° to 360°.

Reflection

$y = \ -f(x)$ ⇒ Reflect in the x-axis

Example: given $y = k^x$, draw $y = \ -k^x$.

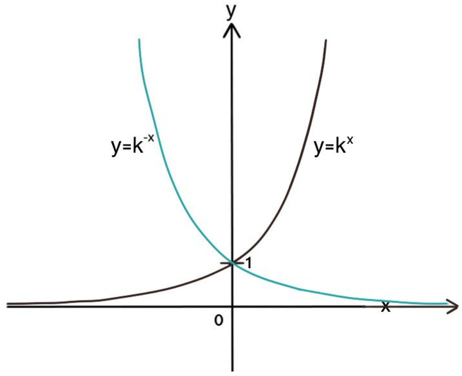


The entire graph is reflected in the x-axis. At each value of x, the value of y has the same magnitude but switches signs i.e. when x=0, y= -1 instead of y= -1.

$y = f(-x)$ ⇒ Reflect in the y-axis

Example: given $y = k^x$, draw $y = k^{-x}$.

The entire graph is reflected in the y-axis. At each value of y, the value of x has the same magnitude but switches signs i.e. when y=k, x= -1 instead of x= -1.



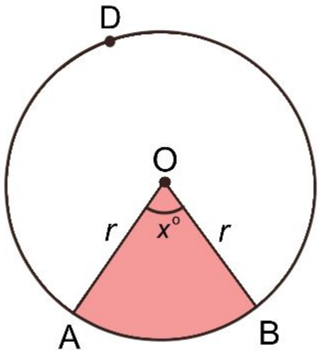
Summary of transformations

$y = f(x) + c$	Translate by c units along the y-axis
$y = k \times f(x)$	Scale by a factor of k along the y-axis
$y = f(x + c)$	Translate by $-$c units along the x-axis
$y = f(kx)$	Scale by a factor of $\frac{1}{k}$ along the x-axis
$y = -f(x)$	Reflect in the x-axis
$y = f(-x)$	Reflect in the y-axis

Area

Circle

- **Sector** is a part of a circle enclosed by any two radii of a circle and an arc
- **Minor sector** is the shaded region enclosed by the radii OA, OB and the minor arc AB
- **Major sector** is the region enclosed by the radii OA, OB and the major arc ADB



- Area of circle $= \pi r^2$
- Circumference of circle $= 2\pi r$
- Area of sector $= \frac{angle\ of\ sector}{360°} \times area\ of\ circle = \frac{x}{360} \times \pi r^2$
- Length of arc $= \frac{angle\ of\ sector}{360°} \times circumference\ of\ circle = \frac{x}{360} \times 2\pi r$

Formulas

- Area of triangle
- $= \frac{1}{2} \times base\ \times vertical\ height$
- Area of quadrilateral $= base\ \times vertical\ height$
- Area of trapezium $= \frac{1}{2} \times sum\ of\ both\ sides\ \times vertical\ height$
- Know how to find the area of composite shapes

Areas of similar figures

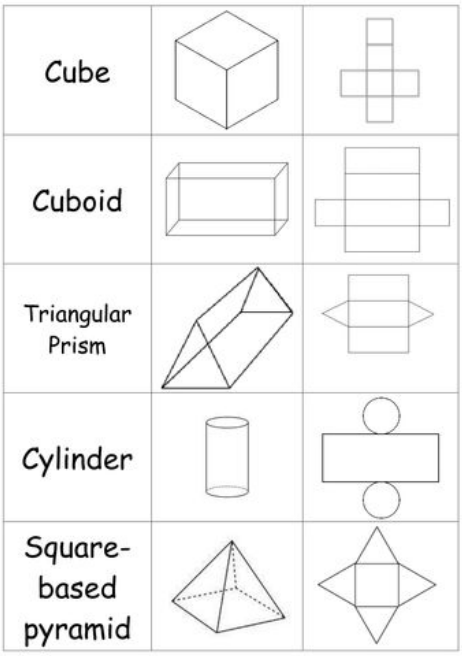
The ratio of the areas of two similar figures is equal to the square of the ratio of any two corresponding lengths of the figures.

If A_1 and A_2 denote the areas of two similar figures, and l_1 and l_2 denote their corresponding lengths $\Rightarrow \frac{A_1}{A_2} = (\frac{l_1}{l_2})^2$

Surface area

- Know that surface area of **sphere** $= 4\pi r^2$
- Know that surface area of **cone** $= \pi rl + \pi r^2$
- Know that surface area of **cylinder** $= 2\pi rh + 2\pi r^2$

Nets of shapes



Volume

- Volume of **cuboid** $= length \times width \times height$
- Volume of **prism** $= cross - sectional\ area \times length$
- Volume of **sphere** $= \frac{4}{3}\pi r^3$
- Volume of **pyramid** $= \frac{1}{3} \times base\ area \times height$
- Volume of **cone** $= \frac{1}{3} \times \pi r^2 \times h$
- Volume of **frustum** $= volume\ of\ cone\ - volume\ of\ removed\ cone$

Volume of similar solids

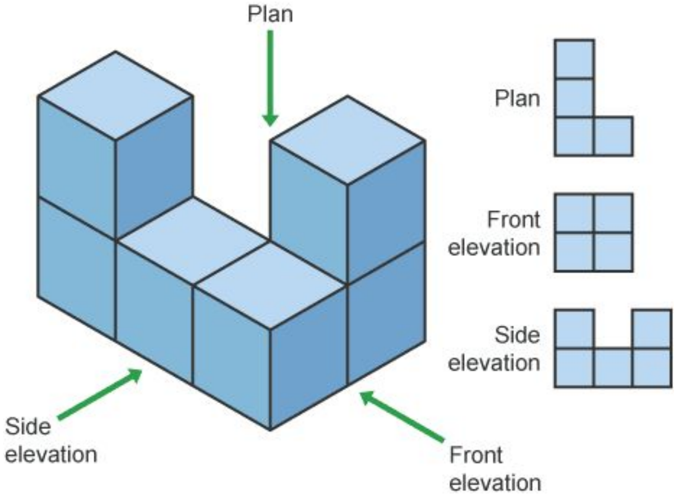
The ratio of the volumes of two similar figures is equal to the cube of the ratio of any two corresponding lengths of the figures.

If V_1 and V_2 denote the areas of two similar figures, and l_1 and l_2 denote their corresponding lengths $\Rightarrow \frac{V_1}{V_2} = (\frac{l_1}{l_2})^3$

Density and speed

- $density = \frac{mass}{volume}$
- $speed = \frac{distance}{time}$

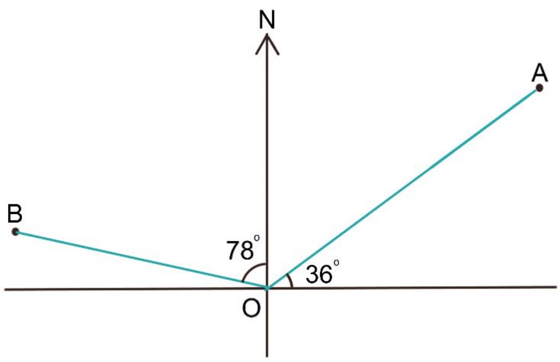
Projections



Bearings

Bearing is an angle measured from the **north**, at O, in a **clockwise direction** to a point and is always written as a **three-digit number**.

Worked example



Find the bearing of i) point A from O ii) point B from O.

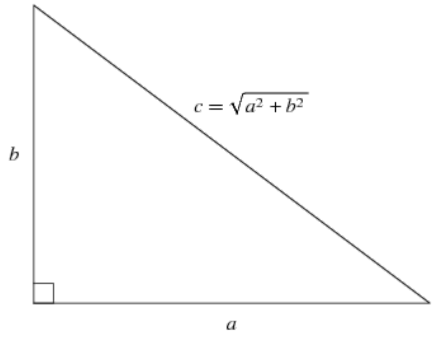
i) Bearing of point A $=$ acute $\angle NOA = 90° - 36° = 054°$

ii) Bearing of point B $=$ obtuse $\angle NOB = 360° - 78° = 282°$

Pythagoras theorem

We can use this theorem to **calculate the lengths of the sides of a right angled triangle**.

In a right angled triangle, the **hypotenuse** (often denoted as c) is the longest side - the one which is opposite the right angle. It does not matter which of the short lengths you call a or b. You can see this clearly in the triangle below:



$a^2 + b^2 = c^2$

Where c is the length of the hypotenuse

Example 1:

$a^2 + b^2 = c^2$

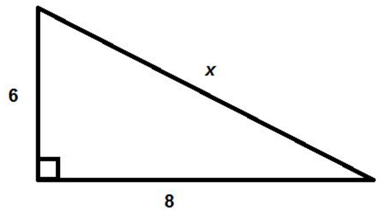
$a = 8 \quad b = 6 \quad c = x$

$8^2 + 6^2 = x^2$

$x^2 = 64 + 36$

$x^2 = 100$

$x = 10$



Example 2:

$a^2 + b^2 = c^2$

$a = 5 \quad b = ? \quad c = 13$

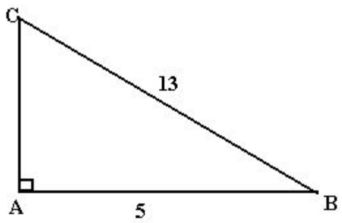
$5^2 + b^2 = 13^2$

$b^2 = 13^2 - 5^2$

$b^2 = 169 - 25$

$b^2 = 144$

$b = 12$



Trigonometry

Basic trigonometric functions

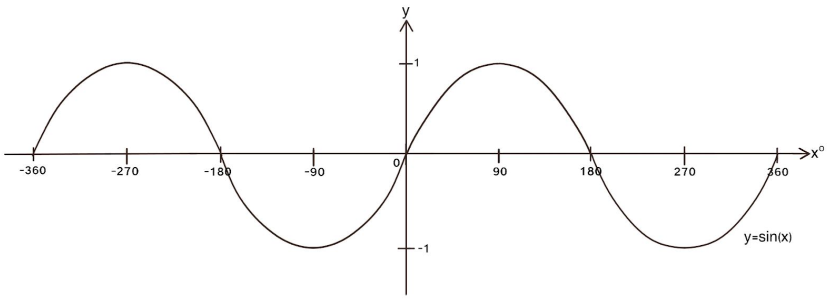
$$sin(x) = \frac{opposite}{hypotenuse}$$
$$cos(x) = \frac{adjacent}{hypotenuse}$$
$$tan(x) = \frac{opposite}{adjacent}$$
$$tan(x) = \frac{sin(x)}{cos(x)}$$

Common values

x (°)	$sin\ (x)$	$cos\ (x)$	$tan\ (x)$
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	1	0	undefined

Sine graphs

Equation: $y = sin\ (x)$



This is the typical appearance of a **sine graph** with the range for y being -1 to 1.

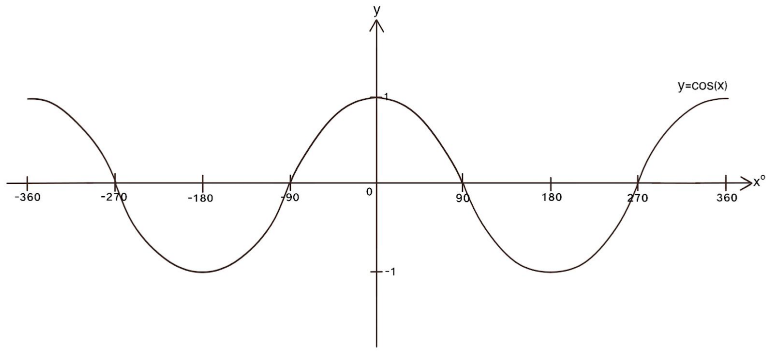
Some sine graphs may look different, for example, having a different range for y, or the peak occurs at $x = 0$ instead of at $x = 90$. But the fundamental shape is still this **wave-like** form and it is just that the sine graph has undergone some transformations which will be addressed in the next section.

If you happen to forget where the peaks occur, write down the common values of $sin\,(x)$ and plot accordingly.

$x\,(°)$	$sin\,(x)$
0	0
30	$\frac{1}{2}$
45	$\frac{\sqrt{2}}{2}$
60	$\frac{\sqrt{3}}{2}$
90	1

Cosine graphs

Equation: $y = cos\,(x)$



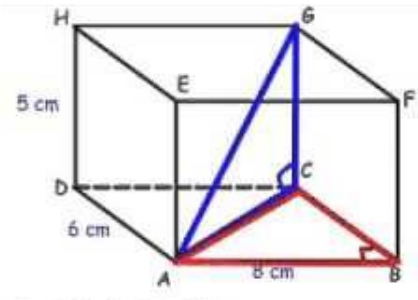
This is the typical appearance of a **cosine graph** with the range for y being -1 to 1.

Some cosine graphs may look different, for example, having a different range for y, or the peak occurs at $x = 90$ instead of at $x = 0$. But the fundamental shape is still this wave-like form and it is just that the cosine graph has undergone some transformations which will be addressed in the next section.

If you happen to forget where the peaks occur, write down the common values of $cos\,(x)$ and plot accordingly.

$x\,(°)$	$cos\,(x)$
0	1
30	$\frac{\sqrt{3}}{2}$
45	$\frac{\sqrt{2}}{2}$
60	$\frac{1}{2}$
90	0

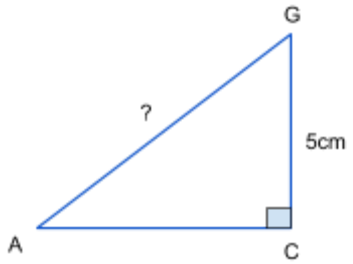
3D Pythagoras and trigonometry



We can use these exact same theorems and values with 3D objects.

Example 1: Calculate the length AG

In this cuboid we can see that the length AG creates a hypotenuse for an imaginary triangle ACG (drawn in blue).
We know the length CG, as it is the same as DH, so 5cm.
However, we will also need the length of AC to calculate the hypotenuse AG.
Therefore, we can create a second triangle on the base of the cuboid (drawn in red).



We can work out AB by using Pythagorus theorem:

$a^2 + b^2 = c^2$

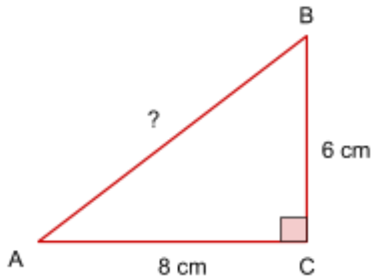
$a = 8 \quad b = \ 6 \quad c = AB$

$8^2 + 6^2 = AB^2$

$AB^2 = 64 + 36$

$AB^2 = \ 100$

$AB = 10$



We can now put this figure into our blue triangle and use Pythagorus theorem again to work out AG:

$a^2 + b^2 = c^2$

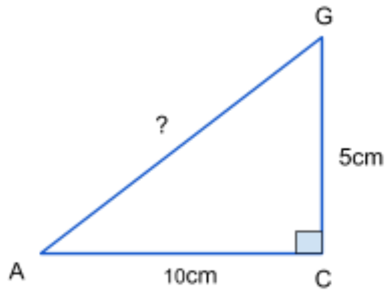
$a = 5 \quad b = \ 10 \quad c = AH$

$5^2 + 10^2 = AH^2$

$AH^2 = 25 + 100$

$AH^2 = \ 125$

$AH = \sqrt{125} = \ 11.2cm$



Example 2: Calculate angle ∠GAC

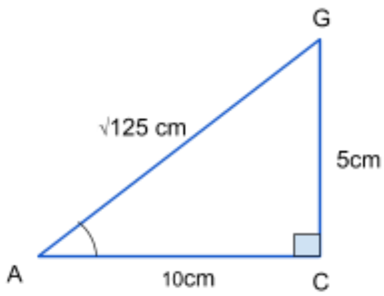
Using the now completed blue triangle, we can use trigonometry to calculate ∠GAC.

We already know these three trigonometric ratios:

$sin(x) = \frac{opposite}{hypotenuse}$

$cos(x) = \frac{adjacent}{hypotenuse}$

$tan(x) = \frac{opposite}{adjacent}$



We can use any of them to calculate ∠GAC

However, $tan(x)$ is probably the easiest to calculate.

$tan(x) = \frac{5}{10} = 0.5 \qquad tan^{-1}(0.5) = 26.6°$

You can check this angle by using the other trigonometric functions.

Vectors

Vector notation

Vectors are used to describe how to move from one point to another. They are different from the measurement of distance, as they describe both **direction** as well as **size**.

They are usually shown by bold letters, e.g. the vector ***a*** (or <u>a</u> if you are handwriting the vector).

However, they are also sometimes written as $\overrightarrow{ab}$

Column vectors

Column vectors represent movement in the x, y and sometimes z direction.

$\begin{pmatrix} a \\ b \end{pmatrix}$ means that you move a places along the x-axis and b across the y-axis.

Adding and subtracting:

You can easily add or subtract across the rows

$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

Multiplying by a scalar:

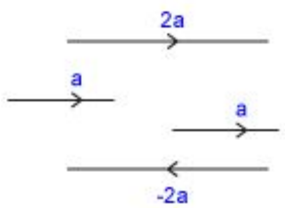
A scalar has magnitude but no direction. For example, speed is scalar (as it does not specify in which direction the movement is occuring). Velocity is a vector because it describes both the speed and the movement.

Therefore, multiplying by a scalar is simple:

If **x** = $\begin{pmatrix} a \\ b \end{pmatrix}$ then 3***x*** $= 3\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix}$

Vectors as diagrams

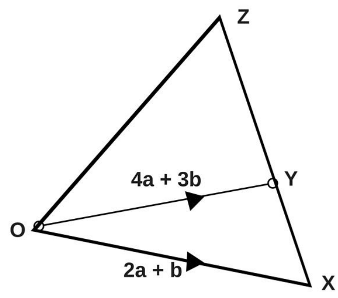
Let's imagine that vector ***a*** is represented as the line which goes from O to A. (O being the (0,0), the origin). This means that any lines in the same direction (i.e. from O to A) and also the same length (as vectors describe both direction and size), must also be equal to ***a.***



However, a vector travelling in the same direction (O to A) but double the length, can be represented by *2**a***.

If the vector was double the size and was travelling from A to O, then we can write this as *-2**a.***

We can use the information we have learnt so far to work through problems using vector diagrams.



Example: Calculate $\overrightarrow{XY}$

Another way of writing $\overrightarrow{XY}$ is to go from $\overrightarrow{XO}$ to $\overrightarrow{OY}$

Therefore:

$\overrightarrow{XY} = \overrightarrow{XO} + \overrightarrow{OY}$

$\overrightarrow{OX}$ = 2a+b and as we are going in the opposite direction, we can say that

$\overrightarrow{XO}$ = -2a-b

Therefore:

$\overrightarrow{XY}$ = -2a - b + 4a + 3b

$\overrightarrow{XY}$ = 2a + 2b