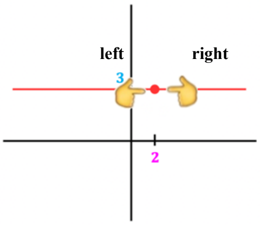
1 Limits

1.1 Graphically

The easiest way to get an intuition about limits is to first look at them graphically.

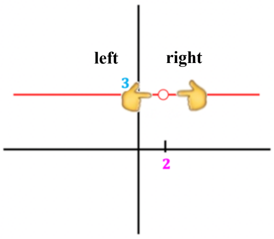
Informally we say, a limit exists at a point if we can trace THE CURVE (the function) inwards with our fingers from either side of the point (from both the left and right of the x value that we are finding the limit at) and tend towards the same y value (reach the same height on the y axis)

Let's look at the limits of the following functions to demonstrate this idea:



This limit exists at $x = 2$

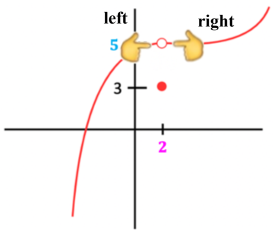
The value of the limit is 3 from both the right and the left and we say the limit is equal to 3



This limit exists at $x = 2$

The value of the limit is 3 from both the right and the left and we say the limit is equal to 3

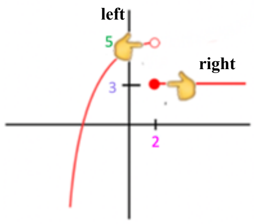
Note: Notice how we don't know the value of the function at $x = 2$ since the circle is not filled in (unlike the example on the left). This doesn't matter. All we can say is that as we approach 2, the limit is 3. So, we can ignore what happens when we get there (i.e.at $x = 2$).



This limit exists at $x = 2$

The value of the limit is 5 from both the right and the left and we say the limit is equal to 5.

As we get closer and closer to $x = 2$ the answer gets closer and closer to 5.
Note: The value of the function at a point does not necessarily have to be equal to the value of the limit at that point. Notice how here we know the value of the function at $x = 2$, but is not equal to value of the limit. The value of the function at $x = 2$ is 3 and the value of the limit at $x = 2$ is equal to 5. This doesn't matter.

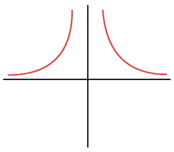


This limit DOES NOT exist at $x = 2$

The value of the limit from the right is equal to 3 and the value of the limit from the left is equal to 5

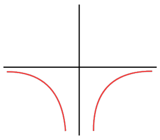
These values are not the same (not equal), so the limit DOES NOT exist.

Let's look at some slightly harder examples where the limit does not exist



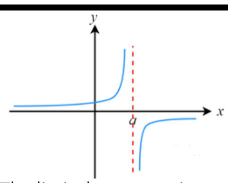
This limit DOES NOT exist at $x = 0$

The limit diverges to ∞ (even though the limits are equal meaning they are both tend to infinity and hence the same from the left and right there is no limit since they are **infinite** limits)



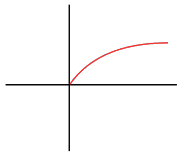
This limit DOES NOT exist at $x = 0$

The limit diverges to $-\infty$ (even though the limits are equal meaning they both tend to $-$ infinity and hence are the same from the left and right there is no limit since they are **infinite** limits)



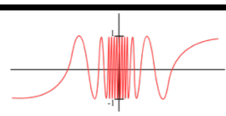
The limit does not exist at $x = a$

The limit from the right is $-\infty$ and the limit from the left is ∞.



The limit DOES NOT exist at $x = 0$

The limit from the right is 0, but the function isn't defined for values to the left of $x = 0$ meaning there are no values to the left of $x = 0$ so there is no limit from the left. Thus, we can't take a limit from that side.

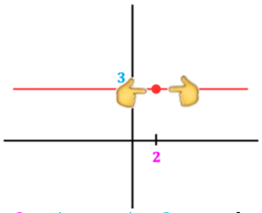


The limit DOES NOT exist at $x = 0$

This is because of oscillatory behavior. The graph of the function oscillates infinitely up and down as x approaches 0. $f(x)$ oscillates between -1 and 1.

Notation

We also have a special notation to talk about limits. Let's consider our very first example.



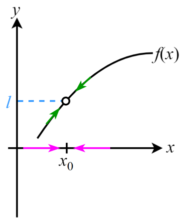
We have already seen that this limit of the function $f(x)$ exists at $x = 2$ and is equal to 3. More formally we can write this as

$$\lim_{x\to 2} f(x) = 3$$

In English this says:

- The symbol lim means we're taking a limit of something. When you see "limit", think "**approaching**". A limit is the value that a function approaches as the x value approaches some value.
- The expression $f(x)$ to the right of lim is the expression we're taking the limit of. In our case, that's the function f
- The expression $x \to 2$ that comes below lim means that we take the limit of the function f as values of x approaches 2.

More generally, we say, a limit exists at a point x_0 and is equal to l if we can trace the graph of $f(x)$ inwards from either side of the point which has an x coordinate of x_0 and tend towards the same y value/height which is l



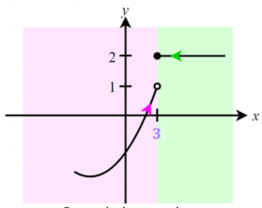
We write this as

$$\lim_{x\to x_0} f(x) = l$$

In English this says that as x approaches x_0 (from the left or the right side of x_0 of the function $f(x)$), the function approaches a y value of l.

One Sided Limits Versus Two Sided Limits

Tracing the curve from EITHER SIDE (one side at a time i.e. from the left or from the right) are called **one-sided** limits. So, a one-sided limit is just the value the function approaches as the x-values approach the limit from ONE SIDE only (left or right). Let's look at an example.



The green region represents the part of the graph to the right of the point $x = 3$ and the pink region represents the part of the graph to the left of this point. We use **superscripts** (powers which are either plus or minus) to indicate these regions:

- $\lim_{x\to 3^-} f(x)$ is called a left limit or left-hand limit or left-handed limit. The **negative superscript** signifies from the **left side** (from x values **less than** x_0). This means we trace/approach the graph from the **left side** i.e. as x approaches 3 from values **less than** 3.

$$\lim_{x\to 3^-} f(x) = 1$$

- $\lim_{x\to 3^+} f(x)$ is called a right limit or right-hand limit or right-handed limit. The **positive superscript** signifies this from the **right side** (from x values **greater than** x_0). This means we trace/approach the graph from the **right side** i.e. as x approaches 3 from values **greater than** 3

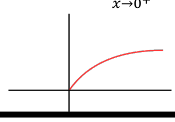
$$\lim_{x\to 3^+} f(x) = 2$$

Note: our familiar $\lim_{x\to 3} f(x)$ is called a two-sided limit. It is basically the limit tested from **both** directions

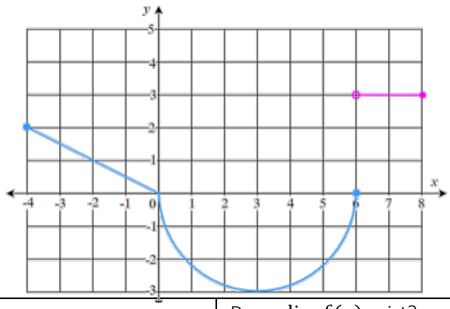
If both of the one-sided limits are not equal then the two-sided limit does not exist. However, if both of the one-sided limits are equal , then the two-sided limit $\lim_{x\to 3} f(x)$ exists. If this two-sided limit exists, we call it l and write the two sides limit as $\lim_{x\to x_0} f(x) = l$.

These one-sided limits are not equal since $1 \neq 2$. Hence the two-sided limit $\lim_{x\to 3} f(x)$ is undefined and we say the limit does not exist

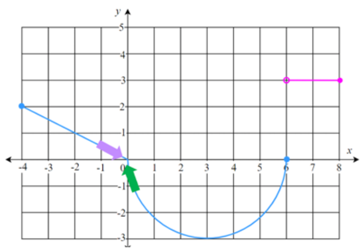
Note:
Sometimes a function is only defined on one side, and in that case, a one-sided limit is the best we can do.
For example, we can only talk about the right sided limit at $x = 0$ for the function below. $\text{Lim}_{x\to 0^+} f(x) = 0$ but $\lim_{x\to 0^-} f(x)$ does not exist since there is no function on that side.



The graph below shows $f(x)$



Does $\lim_{x\to 0} f(x)$ exist?

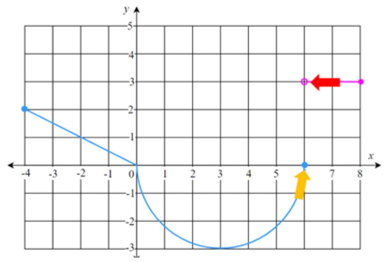


$\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

Yes, the limit does exist at $x = 0$ since the limits both are the same (they both are equal to zero)

Does $\lim_{x\to 6} f(x)$ exist?

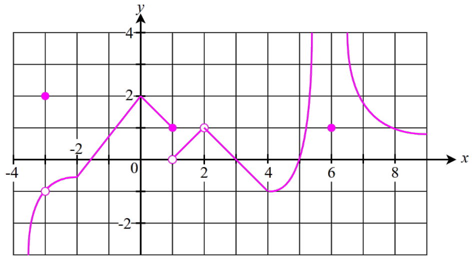


$\lim_{x\to 6^+} f(x) = 3$

$\lim_{x\to 6^-} f(x) = 0$

No, the limit does not exist at $x = 6$ since the limits are not equal ($3 \neq 0$)

1) The graph below shows the graph of $f(x)$



a) Does $\lim_{x\to -3} f(x)$ exist? If so, state what it is.

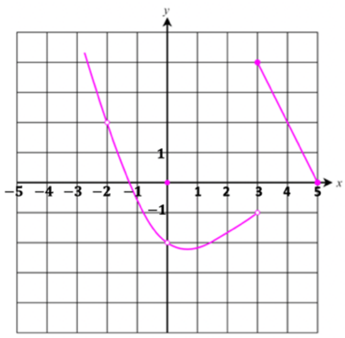
b) Does $\lim_{x\to 1} f(x)$ exist? If so, state what it is.

c) Does $\lim_{x\to 2} f(x)$ exist? If so, state what it is.

d) Does $\lim_{x\to 4} f(x)$ exist? If so, state what it is.

e) Does $\lim_{x\to 6} f(x)$ exist? If so, state what it is.

2)



Find

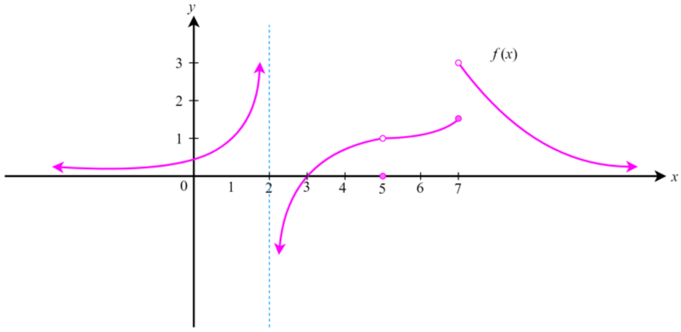
i. $\lim_{x\to -2} f(x)$

ii. $\lim_{x\to 0} f(x)$

iii. $\lim_{x\to 3} f(x)$

iv. $\lim_{x\to 5^-} f(x)$

3)



i. $\lim_{x \to 5} f(x)$

ii. $\lim_{x \to 2^+} f(x)$

iii. $\lim_{x \to 2^-} f(x)$

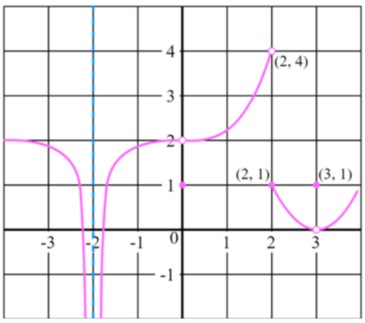
iv. $\lim_{x \to 7^+} f(x)$

v. $\lim_{x \to 7^-} f(x)$

vi. $\lim_{x \to 7} f(x)$

vii. $\lim_{x \to \infty} f(x)$

4)



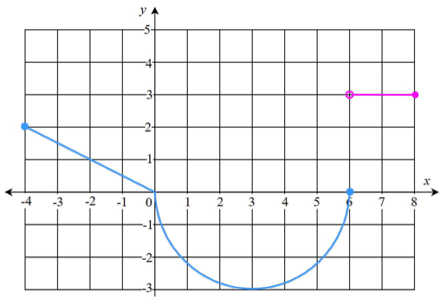
Find

i. $f(0)$

ii. $\lim_{x \to 2^-} f(x)$

iii. $\lim_{x \to 3} f(x)$

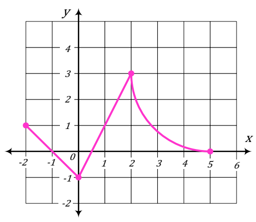
5) The graph below shows the graph of $f(x)$



Find $\lim_{x \to -2} \frac{f(x)+7}{e^{3x+6}-1}$

6)

t (hours)	0	0.3	1	2.8	4
$v_p(t)$ (meters per hour)	0	55	-29	55	48



Graph of f

Find $\lim_{t \to 1} \frac{e^t - 3f(t)}{v_p(t) - \cos(\pi t)}$

1.2 Algebraically

For the easiest types of questions we can simply substitute the numbers into the function and we are done. Let's see how this work with a few basic examples.

$\lim_{x\to 3} 2x$ Let's colour code for ease of explanation $\lim_{x\to 3} 2x$ This tells us to replace x with 3 in the expression $2x$ Substituting gives $2(3)$ $= 6$	$\lim_{x\to 2}(x^2+5x)$ Let's colour code for ease of explanation $\lim_{x\to 2}(x^2+5x)$ This tells us to replace x with 2 in the expression x^2+5x Substituting gives $(2^2+5(2))$ $= 14$	$\lim_{x\to 0}\frac{x^2+4}{x-2}$ Let's colour code for ease of explanation $\lim_{x\to 0}\frac{x^2+4}{x-2}$ This tells us to replace x with 0 in the expression $\frac{x^2+4}{x-2}$ Substituting gives $\frac{0^2+4}{0-2}$ $= -2$	$\lim_{x\to 2}(2x-1)^4$ Let's colour code for ease of explanation $\lim_{x\to 2}(2x-1)^4$ This tells us to replace x with 2 in the expression $(2x-1)^4$ Substituting gives $(2(2)-1)^4$ $= 3^4$ $= 81$	$\lim_{x\to \pi}(3\sin x-2x)$ Let's colour code for ease of explanation $\lim_{x\to \pi}(3\sin x-2x)$ This tells us to replace x with π in the expression $(3\sin x-2x)$ Substituting gives $(3\sin\pi-2\pi)$ $= 3(0)-2(\pi)$ $= -2\pi$

For all the above examples we say the limit exists and whatever number we get is the value of the limit. However, we don't always get a non-zero numbers or 'nice' numbers. Sometimes we get zero and undefined answers. Let's look at a few examples

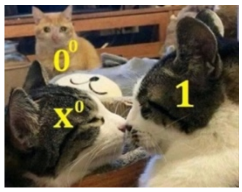
$\lim_{x\to\infty}\frac{x}{2}$ Substitute $x=\infty$ $\lim_{x\to\infty}\frac{x}{2}$ $=\frac{\infty}{2}$ A very very big number over a much smaller negligible number in comparison will always remain a very very big number $=\infty$ We say the limit **doesn't** exist. We call this undefined.	$\lim_{x\to 0}\frac{x}{5}$ Substitute $x=0$ $\lim_{x\to 0}\frac{x}{5}$ $=\frac{0}{5}$ Zero divided by a non-zero number is always 0 $=0$ We say the limit exists and is equal to zero	$\lim_{x\to\infty}\frac{3}{x}$ Substitute $x=\infty$ $\lim_{x\to\infty}\frac{3}{x}$ $=\frac{3}{\infty}$ If we divide by a very very large number we are practically 0. Think about it. The more slices you cut a cake into the smaller the slices become. If we kept cutting the cake size would get smaller and smaller until eventually we would barely get any cake. So here we get $=0$ We say the limit exists and is equal to zero

Let's summarise this

$\frac{\text{any non zero number}}{0}=\text{undefined}$ We say the limit doesn't exist	$\frac{\pm\infty}{\text{any non infinte number}}=\pm\infty$ We say the limit doesn't exist	$\frac{0}{\text{any nonzero number}}=0$ We say the limit exists and is equal to zero	$\frac{\text{any non infinite number}}{\pm\infty}=0$ We say the limit exists and is equal to zero

Unfortunately, we can also get worse than this. We can substitute and get one of the following **7 indeterminate forms:**

$$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad \infty-\infty \qquad 0(\infty) \qquad 0^0 \qquad 1^\infty \qquad \infty^0$$



It becomes a process of elimination of what to do next when this occurs. We must use one of the following 7 methods on the following pahge (we try in the order of left to right and see which works) and then substitute again

Do I have a fraction and can I factorise and cancel?

Your factorising needs to be good (see my cheat sheet if not)

Factorise and cancel

Example 1 :

$$\lim_{x\to 2}\frac{x^2-4x+4}{x-2}$$

substitution yields $\frac{0}{0}$

Let's factorise first:

$$\lim_{x\to 2}\frac{(x-2)(x-2)}{x-2}$$

$$=\lim_{x\to 2}(x-2)$$

Now we can substitute

$$2-2$$

$$=0$$

Example 2 :

$$\lim_{x\to -1}\frac{x^2-x-2}{x^2-4x-5}$$

substitution yields $\frac{0}{0}$

Let's factorise first:

$$\lim_{x\to -1}\frac{(x-2)(x+1)}{(x-5)(x+1)}$$

$$=\lim_{x\to -1}\frac{x-2}{x-5}$$

Now we can substitute

$$=\frac{-1-2}{-1-5}$$

$$=\frac{-3}{-6}=\frac{1}{2}$$

Do I have a fraction with a square root in the numerator or denominator? Rationalise!

Rationalise the numerator or denominator and cancel

Example:

$$\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$$

substitution yields $\frac{0}{0}$

Let's rationalise the numerator

Consider the function

$$\frac{\sqrt{x+5}-3}{x-4}\times\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$$

$$\lim_{x\to 4}\frac{x+5-9}{(x-4)(\sqrt{x+5}+3)}$$

$$=\lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x+5}+3)}$$

Now we can cancel

$$=\lim_{x\to 4}\frac{1}{\sqrt{x+5}+3}$$

$$=\frac{1}{3+3}$$

$$=\frac{1}{6}$$

Do I have a fraction and can I differentiate the numerator and denominator?

Note: You need to have $\frac{0}{0}$ to be able to do this

You can differentiate any number of times!

Apply **L' Hopital's Rule**

It is important to know that you **can only** apply L' Hopital's Rule as long as you have an indeterminate form and you can apply this as many times as you want! We just differentiate the numerator and denominator.

Example:

$$\lim_{x\to -3}\frac{x^3+9x^2+27x+27}{x^3+8x^2+21x+18}$$

substitution yields $\frac{0}{0}$

Use L'Hopital's (differentiate the numerator and denominator)

$$\lim_{x\to -3}\frac{3x^2+18x+27}{3x^2+16x+21}$$

substitution still yields $\frac{0}{0}$

Use L'Hopital's again

$$\lim_{x\to -3}\frac{6x+18}{6x+16}$$

Substitute

$$=\frac{6(-3)+18}{6(-3)+16}$$

$$=\frac{0}{-2}=0$$

Do I have trig? Can I use the identities:

$$\sin^2 x+\cos^2 x=1$$
$$1+\tan^2 x=\sec^2 x$$
$$1+\cot^2 x=\text{cosec}^2 x$$
$$\sin 2x=2\sin x\cos x$$
$$\cos 2x=\cos^2 x-\sin^2 x$$
$$\sin(a\pm b)=\sin a\cos b\pm\cos a\sin b$$
$$\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$$

If trig we use a **trig identity** and maybe simplify after

Example:

$$\lim_{x\to\pi}\frac{\tan^2 x}{1+\sec x}$$

substitution yields $\frac{0}{0}$

Let's use a trig identity first in the numerator

$$\lim_{x\to\pi}\frac{\sec^2 x-1}{1+\sec x}$$

Factorise (difference of two squares)

$$\lim_{x\to\pi}\frac{(\sec x+1)(\sec x-1)}{1+\sec x}$$

$$=\lim_{x\to\pi}(\sec x-1)$$

$$=\sec\pi-1$$

$$=\frac{1}{\cos\pi}-1$$

$$=-1-1$$

$$=-2$$

Do I have trig and is my limit tending to zero

$$\sin\theta\approx\theta$$
$$\cos\theta\approx 1-\frac{\theta^2}{2}$$
$$\tan\theta\approx\theta$$

Use **small angle approximations**

We **can only do this** if trig and $\to$ **0**. It is important that you realise this! As soon as a limit tends to zero for trig, you should check first whether small angle approximations will help!

Example:

$$\lim_{x\to 0}\frac{\sin x}{x}$$

substitution yields $\frac{0}{0}$

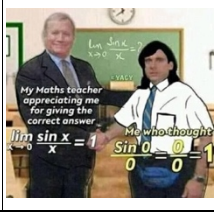
Use small angle approx.

$$\lim_{x\to 0}\frac{x}{x}$$

$$\lim_{x\to 0}1$$

$$=1$$

when $\sin\theta=\theta$:



Is x tending to infinity?

Do I get $\frac{\pm\infty}{\pm\infty}$ when I substitute?

Divide the numerator and denominator by the highest power in the denominator and then the "bottom-heavy" terms tend to zero (the terms with a number in the numerator and x, x^2 etc in the denominator).

See the 4 columns below for detailed examples on this.

Have the 6 methods to the left failed? Use squeeze theorem! This is for $x\to 0$ and $x\to\infty$ (this type usually occurs for trig , factorials or powers of x)

Squeeze theorem!

This basically says if we can find a smaller and bigger function that have the same limit, then the function "squeezed between them" must also have the same limit.

Example 1:

$$\lim_{x\to 0}x\cos\left(\frac{1}{x}\right)$$

Substitution gives $0\times\infty$

Instead us squeeze theorem

Compare to $-|x|$ *and* $|x|$

$$\lim_{x\to 0}-|x|=0$$
$$\lim_{x\to 0}|x|=0$$

$$-|x|\le x\cos\left(\frac{1}{x}\right)\le|x|$$

since $-1\le\cos\left(\frac{1}{x}\right)\le 1$

$$\lim_{x\to 0}x\cos\left(\frac{1}{x}\right)=0$$

Example 2:

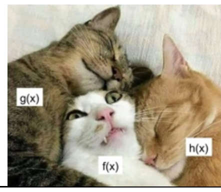
$$\lim_{x\to\infty}\frac{\sin x}{x}$$

Substitution gives $\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right)=0$$
$$\lim_{x\to\infty}\left(\frac{1}{x}\right)=0$$
$$-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$$

By squeeze theorem

$$\lim_{x\to\infty}\left(\frac{\sin x}{x}\right)=0$$



Limits tending to infinity:

Bottom-heavy (denominator has the highest power)

$$\lim_{x\to\infty}\frac{2x+3}{4x^2-5}$$

Highest power in denominator is a 2 due to x^2 term, so let's divide all terms by this

$$\lim_{x\to\infty}\frac{\frac{2x}{x^2}+\frac{3}{x^2}}{\frac{4x^2}{x^2}-\frac{5}{x^2}}=\lim_{x\to\infty}\frac{\frac{2}{x}+\frac{3}{x^2}}{4-\frac{5}{x^2}}$$

Substitute and get $\frac{0+0}{4-0}$=0

Shortcut method:

Here the highest power in the denominator is greater than that of the numerator hence bottom-heavy (the bottom becomes very large) and the limit is zero

Even powers (tie with highest power of numerator & denominator)

$$\lim_{x\to\infty}\frac{5x^2+5x-5}{6x^2-2x+5}$$

Highest power in denominator is a 2 due to x^2 term, so let's divide all terms by this

$$\lim_{x\to\infty}\frac{\frac{5x^2}{x^2}+\frac{5x}{x^2}-\frac{5}{x^2}}{\frac{6x^2}{x^2}-\frac{2x}{x^2}+\frac{5}{x^2}}=\lim_{x\to\infty}\frac{5+\frac{5}{x}-\frac{5}{x^2}}{6-\frac{2}{x}+\frac{5}{x^2}}$$

Substitute and get $\frac{5+0-0}{6-0+0}=\frac{5}{6}$

Shortcut method:

Here the ratio of the highest power terms which are x terms is $\frac{3}{2}$. Hence limit $=\frac{3}{2}$

Top-heavy (numerator has the highest power)

$$\lim_{x\to\infty}\frac{10x^2+x}{4x-1}$$

Highest power in denominator is a 1 due to x term, so let's divide all terms by this

$$\lim_{x\to\infty}\frac{\frac{10x^2}{x}+\frac{x}{x}}{\frac{4x}{x}-\frac{1}{x}}=\lim_{x\to\infty}\frac{10x+1}{4-\frac{1}{x}}$$

Substitute and get $\frac{\infty}{4-0}=\infty$

Shortcut method:

Here the highest power in the numerator is greater than that of the denominator hence top-heavy and hence there is no limit (infinite)

Careful if you have a root in the denominator. If $x\longrightarrow\infty$ divide inside root by x^2 and outside by x

$$\lim_{x\to\infty}\frac{2x}{\sqrt{4x^2+1}}$$

$$\lim_{x\to\infty}\frac{\frac{2x}{x}}{\sqrt{\frac{4x^2}{x^2}+\frac{1}{x^2}}}$$

$$\lim_{x\to\infty}\frac{2}{\sqrt{4+\frac{1}{x^2}}}$$

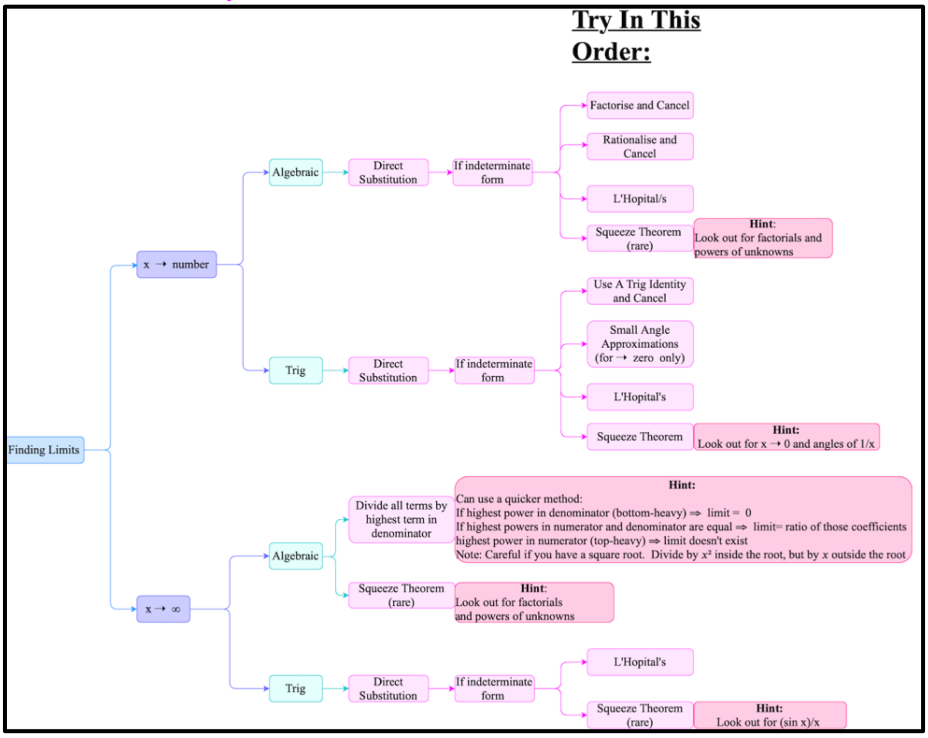
Substitute

$$=\frac{2}{\sqrt{4+0}}$$

$$=\frac{2}{2}=1$$

We can apply a shortcut for these types. If the

- numerator wins, then the limit will be $y=0$
- denominator wins, the limit will be $y=\pm\infty$
- There is a tie, the limit will be a finite number which is the ratio of terms with those coefficients of terms with highest power



1.3 Tending To A Number

Easy

1) $\lim_{x\to 2} x^3$
2) $\lim_{x\to 0}(2x-1)$
3) $\lim_{x\to 7}(2x-16)^4$
4) $\lim_{x\to 1}\sqrt{\frac{2x+1}{2x-1}}$
5) $\lim_{x\to 1}\frac{3x+1}{x+1}$
6) $\lim_{x\to 0}\frac{x}{5}$
7) $\lim_{x\to 3}\frac{2x-6}{8}$
8) $\lim_{x\to 5}\frac{x-5}{x+3}$
9) $\lim_{x\to 2}\frac{3x+3}{x-2}$
10) $\lim_{x\to\infty}\frac{3}{x}$
11) $\lim_{x\to\infty}\frac{5}{2x-1}$

Medium

1) $\lim_{x\to 5}\frac{x-5}{x^2-25}$
2) $\lim_{x\to 0}\frac{x}{x^2-x}$
3) $\lim_{x\to 0}\frac{x^4-5x^2}{x^2}$
4) $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1}$
5) $\lim_{x\to 3}\frac{x^2-x-6}{x^2+2x-15}$
6) $\lim_{x\to 1}\frac{2x^2-x+3}{x-1}$
7) $\lim_{x\to 2}\frac{x^3-8}{x-2}$
8) $\lim_{x\to -3}\frac{2x^2-18}{x+3}$

Hard

1) $\lim_{x\to k}\frac{x^2-k^2}{x^2-kx}, k\neq 0$
2) $\lim_{x\to 0}\frac{\frac{1}{x-1}+1}{x}$
3) $\lim_{x\to 1}\frac{\frac{5x}{2}-\frac{5\pi}{x+1}}{3x-3}$
4) $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$
5) $\lim_{x\to 0}\frac{x}{\sqrt{x+2}-\sqrt{2}}$
6) $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{1-x}$
7) $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$

1.3.1 L'Hopitals

Note: You can use this when other types also work. It just makes your life easier.

Easy

1) $\lim_{x\to 3}\frac{x^2-x-6}{x^2+2x-15}$
2) $\lim_{x\to 1}\frac{x^3+x-2}{x^2-3x+2}$
3) $\lim_{x\to -3}\frac{x^3+9x^2+27x+27}{x^3+8x^2+21x+18}$

Medium
(only attempt this section if you have done advanced differentiation)

1) $\lim_{x\to 0}\frac{2x+x^2+2\ln(1-x)}{x^3}$
2) $\lim_{x\to 0}\frac{tan^3x-\sin^3 x}{x^5}$
3) $\lim_{x\to 0}\frac{tan^2x-\arctan^2 x}{x^4}$

Hard
(only attempt this section if you have done advanced differentiation)

1) $\lim_{x\to\infty} xsin\left(\frac{1}{x}\right)$

Hint: write as $\frac{sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$

1.4 Tending To Infinity

Easy

1) $\lim_{x\to\infty}\frac{x}{2}$
2) $\lim_{x\to\infty}\frac{3x+5}{2x+4}$
3) $\lim_{x\to\infty}\frac{x+8}{x-3}$
4) $\lim_{x\to\infty}\frac{x-5}{7x^2+10x}$
5) $\lim_{x\to\infty}\frac{3-x}{x^2+3}$
6) $\lim_{x\to\infty}\frac{5x^2}{x+4}$
7) $\lim_{x\to\infty}\frac{x^3-27}{x-3}$
8) $\lim_{n\to\infty}\frac{5n^2-4}{n^2-1}$
9) $\lim_{x\to\infty}\frac{5x^3+1}{10x^3-3x^2+7}$
10) $\lim_{x\to\infty}\frac{x^2-6}{2+x-3x^2}$
11) $\lim_{x\to\infty}\frac{4x^2-2x+15}{x^4+6}$

Medium

1) $\lim_{x\to\infty}\frac{x+3}{(5-x)^2}$
2) $\lim_{x\to-\infty}\frac{x+3}{(5-x)^2}$
3) $\lim_{x\to-\infty}\frac{3x^2+1}{(3-x)(3+x)}$

Hard

1) $\lim_{x\to-\infty}\frac{2x}{\sqrt{4x^2+1}}$
2) $\lim_{x\to-\infty}\frac{\sqrt{9x^2+2}}{4x+3}$
3) $\lim_{n\to\infty}\frac{2^n+3^n}{2^n+(-4)^n}$

1.5 Trig

Easy

1) $\lim_{x\to\frac{\pi}{2}}\sin x$
2) $\lim_{x\to1}\cos\frac{\pi x}{3}$
3) $\lim_{x\to0}\sec 2x$
4) $\lim_{x\to\pi}\tan x$
5) $\lim_{x\to\frac{\pi}{6}} sin^3 x sec^4 x$
6)
7) $\lim_{x\to0}\frac{x^2-2}{\cos x}$

Medium

1) $\lim_{x\to0}\frac{\sin 2x}{x}$
2) $\lim_{x\to0}\frac{sin^2 3x}{5x^2}$
3) $\lim_{x\to0}\frac{tan^2 5x}{10x^3}$
4) $\lim_{x\to0}\frac{\cos 3x}{x}$
5) $\lim_{x\to0}\frac{1-cos x}{1-x}$
6) $\lim_{x\to0}\frac{1-cos\, 3x}{5x}$
7) $\lim_{x\to0}\frac{cos2x tan2x}{x}$

Hard

1) $\lim_{x\to\pi}\frac{tan^2 x}{1+sec x}$
2) $\lim_{x\to\frac{3\pi}{2}}\frac{6\cot^2 x}{1+cosec\, x}$
3) $\lim_{x\to0}\frac{\cos\left(x+\frac{\pi}{2}\right)}{2x}$
4) $\lim_{x\to0}\frac{\cos\left(x-\frac{\pi}{3}\right)-\frac{1}{2}}{x}$

1.6 One Sided

These are not as easy to calculate as when we had the graphs.

Example 1: $f(x) = \begin{cases} -3x+7, x \le 2 \\ x^2-5, x > 2 \end{cases}$

Determine $\lim_{x\to 2} f(x)$ if it exists

Here we have two functions. Let's colour code.

$f(x) = \begin{cases} -3x+7, x \le 2 \\ x^2-5, x > 2 \end{cases}$

Once behaves like $-3x+7$ for values of x less than or equal to 2 and the other behaves like x^2-5 for values bigger than 2

We want to find the limit as x tends to 2, but the function behaves differently either side of 2, so we have to find 2 limits (one for just greater than 2 and the other for just less than 2)

$\lim_{x\to 2^+} f(x) = (2)^2 - 5 = -1$

$\lim_{x\to 2^-} f(x) = 3x + 7 = 3(2) - 7 = -1$

Both limits equal -1, so the limit exists

Example 2: Find $\lim_{x\to -1^-} \frac{2x+1}{3x-1}$ and $\lim_{x\to -1^+} \frac{2x+1}{3x-1}$

$\lim_{x\to -1^-} \frac{2x+1}{3x-1} = \frac{2(-1)+1}{3(-1)-1} = \frac{-1}{-4} = \frac{1}{4}$

$\lim_{x\to -1^+} \frac{2x+1}{3x-1} = \frac{2(-1)+1}{3(-1)-1} = \frac{-1}{-4} = \frac{1}{4}$

Example 3: Find $\lim_{x\to 1^+} \frac{x+7}{x-1}$ and $\lim_{x\to 1^-} \frac{x+7}{x-1}$

This is harder since we get $\frac{8}{0}$ for both when we substitute, but we need to know whether our answer is ∞ or $-\infty$.

$\lim_{x\to 1^+} \frac{x+7}{x-1} = \infty$ and $\lim_{x\to 1^-} \frac{x+7}{x-1} = -\infty$

If you're struggling to see why one is plus and one is minus consider the following:

- $\lim_{x\to 1^+}$ means plug in a value just greater than one, say 1.0002 and the denominator is positive hence tends to $+\infty$
- $\lim_{x\to 1^-}$ means plug in a value just less than one, say 0.999999 and the denominator is negative hence tends to $-\infty$

Example 4: Find $\lim_{x\to 2} \frac{|x-2|}{x-2}$

We need to have knowledge of the modulus functions in order to be able to write this as a piecewise function

We can write this as a piecewise function:

$$f(x) = \begin{cases} \frac{x-2}{x-2} = 1, & x \ge 2 \\ \frac{-(x-2)}{x-2} = -1, & x \le 2 \end{cases}$$

Simplifying gives

$$f(x) = \begin{cases} 1, & x \ge 2 \\ -1, & x \le 2 \end{cases}$$

We want to find the limit as x tends to 2, but the function behaves differently either side of 2, so we have to find 2 limits (one for just greater than 2 and the other for just less than 2)

Let's now find the limits

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 1 = 1$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-1) = -1$

One limit is -1 and the other is 1 so limit doesn't exist

Easy

1) $f(x) = \begin{cases} x^2-1, x < 1 \\ x^2+1, x > 1 \end{cases}$
Determine $\lim_{x\to 1} f(x)$ if it exists

2) $g(x) = \begin{cases} \sqrt{2x-1}, x < 1 \\ \frac{x^3+1}{2}, x > 1 \end{cases}$
Find $\lim_{x\to -1} g(x)$

3) $f(x) = \begin{cases} \frac{2x+1}{3x-1}, x \le -1 \\ \frac{1}{(x-1)^2}, x > -1 \end{cases}$
Find $\lim_{x\to -1} f(x)$

4) $h(x) = \begin{cases} 3x^3-2, x < 0 \\ (x+1)^2, x \ge 0 \end{cases}$
Find $\lim_{x\to 0} h(x)$

Medium

1) $\lim_{x\to 2^-} \frac{3x-1}{x+4}$ and $\lim_{x\to 2^+} \frac{3x-1}{x+4}$

2) $\lim_{x\to -3^-} \frac{x+1}{2x-3}$ and $\lim_{x\to -3^+} \frac{x+1}{2x-3}$

3) $\lim_{x\to 3^+} \frac{x+8}{x-3}$ and $\lim_{x\to 3^-} \frac{x+8}{x-3}$

4) $\lim_{x\to -2^+} \frac{x-1}{x+2}$ and $\lim_{x\to -2^-} \frac{x-1}{x+2}$

5) $\lim_{x\to 0^-} \frac{5x-1}{\sqrt{x^2-x}}$ and $\lim_{x\to -1^+} \frac{5x-1}{\sqrt{x^2-x}}$

6) $\lim_{x\to -0.5^-} \frac{\sqrt{9x^2-2}}{2x+1}$ and $\lim_{x\to -0.5^+} \frac{\sqrt{9x^2-2}}{2x+1}$

Hard

1) $\lim_{x\to 0} |x|$

2) $\lim_{x\to 1} |x-1|$

3) $\lim_{x\to 0} \frac{4x}{|x|}$

4) $\lim_{x\to 2} \frac{|x-2|}{x-2}$

1.7 Squeeze Theorem or L'Hopitals

Easy

1) $\lim_{x\to\infty} cos\left(\frac{1}{x}\right)$

2) $\lim_{x\to\infty} sin\left(\frac{1}{x}\right)$

3) $\lim_{x\to\infty} \frac{sinx}{x}$

4) $\lim_{x\to\infty} \frac{sin3x}{x}$

5) $\lim_{x\to\infty} \frac{cosx}{2x}$

Medium

1) $\lim_{x\to\infty} xsin\left(\frac{1}{x}\right)$

2) $\lim_{x\to\infty} xcos\left(\frac{1}{x}\right)$

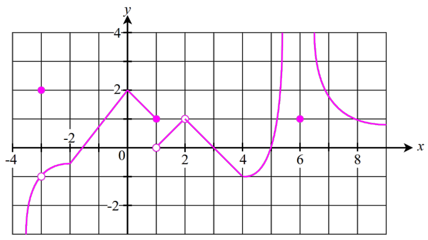
Hard

1) $\lim_{x\to\infty} \frac{x!}{x^x}$

2) $\lim_{x\to\infty} \frac{x^x}{(2x)!}$

Summary – showing when a limit does exist versus doesn't exist

Graphically



- Does $\lim_{x\to -3} f(x)$ exist?

$\lim_{x\to -3^+} f(x) = -1$ and $\lim_{x\to -3^-} f(x) = -1$

$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^-} f(x)$

$\therefore \lim_{x\to -3} f(x)$ exists

- Does $\lim_{x\to 2} f(x)$ exist?

$\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = 1$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$

$\therefore \lim_{x\to 2} f(x)$ exists

- Does $\lim_{x\to 4} f(x)$ exist?

$\lim_{x\to 4^+} f(x) = -1$ and $\lim_{x\to 4^-} f(x) = -1$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x)$

$\therefore \lim_{x\to 4} f(x)$ exists

- Does $\lim_{x\to 1} f(x)$ exist?

$\lim_{x\to 1^+} f(x) = 0$ and $\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

$\therefore \lim_{x\to 1} f(x)$ does not exist

- Does $\lim_{x\to 6} f(x)$ exist?

$\lim_{x\to 6^+} f(x) = \infty$ and $\lim_{x\to 6^-} f(x) = \infty$

$\lim_{x\to 6^+} f(x) \neq \lim_{x\to 6^-} f(x)$

$\therefore \lim_{x\to 6} f(x)$ does not exist

Algebraically

Substitute and show you get any whole number (not ∞ ***or*** $-\infty$)

Example where a limit exists:

$$\lim_{x\to 3} 2x$$

Let's colour code for ease of explanation

$$\lim_{x\to 3} 2x$$

This tells us to replace x with 3 in the expression $2x$

Substituting gives $2(3) = 6$.

So, we get a number which means that the limit exists

Example where a limit doesn't exist:

$$\lim_{x\to 0} \frac{2}{x}$$

Substituting gives $\frac{2}{0}$ which is undefined and therefore the limit doesn't exist

Piecewise Functions - Show 1 sided limits are equal

Example where a limit exists:

$$f(x) = \begin{cases} -3x + 7, x \leq 2 \\ x^2 - 5, x > 2 \end{cases}$$

Determine $\lim_{x\to 2} f(x)$ if it exists

Here we have two functions.

$$f(x) = \begin{cases} -3x + 7, x \leq 2 \\ x^2 - 5, x > 2 \end{cases}$$

$$\lim_{x\to 2^+} f(x) = (2)^2 - 5 = -1$$

$$\lim_{x\to 2^-} f(x) = 3x + 7 = 3(2) - 7 = -1$$

Both limits equal -1, so the limit exists

Example where a limit doesn't exist:

$$\lim_{x\to 2} \frac{|x-2|}{x-2}$$

We need to have knowledge of the modulus functions in order to be able to write this as a piecewise function

We can write this as a piecewise function:

$$f(x) = \begin{cases} \frac{x-2}{x-2} = 1, & x \geq 2 \\ \frac{-(x-2)}{x-2} = -1, & x \leq 2 \end{cases}$$

Simplifying gives

$$f(x) = \begin{cases} 1, & x \geq 2 \\ -1, & x \leq 2 \end{cases}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 1 = 1$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-1) = -1$$

One limit is -1 and the other is 1 so limit doesn't exist