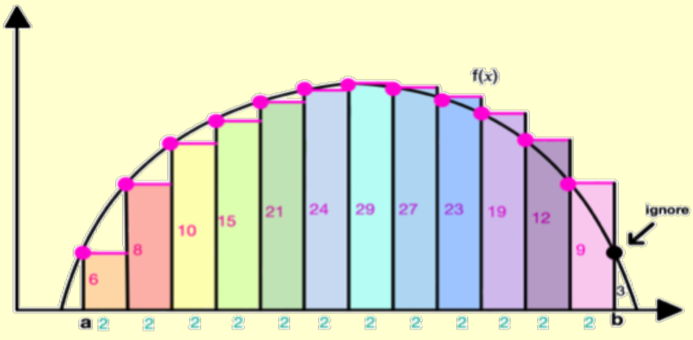
Type	Explanation

Intuition

Instead of directly integrating to find the areas, we can approximate these areas by drawing shapes underneath the curves and adding the areas of all the shapes together. The shapes we commonly draw are rectangles (known as Riemann Sum) or trapezia (known as trapezium rule). We can then add up all the areas of these shapes individually. Remember, definite integrals represent the exact area under a given curve, so it should make total sense that we can consider the shapes underneath the curve and find their areas instead. The more trapezia/rectangles we draw, the better the answer that we get approximates the actual value of $\int_a^b f(x)\,dx$. You will never get the EXACT area but at least it will be more accurate (unless you have a horizontal line with a Riemann sum or diagonal line with trapezium sum then it will be exact)

Left Riemann Sum

Each rectangle's height is determined by evaluating f(x) at the **LEFT**-hand endpoint of the subinterval the rectangle lives on i.e. the height (y coordinate) of each rectangles is determined by **left** endpoint of each x interval for each rectangle shown with the pink point



In a table format this looks like:

X	a												b
Y	6	8	10	15	21	24	29	27	23	19	12	9	3

Area = sum of the areas of all rectangles
= (2x6) +(2x8)+(2x10)+(2x15)+(2x21)+(2x24)+(2x29)+(2x27)+(2x23)+(2x19)+(2x12)+(2x9)
= 2 (6+8+10+15+21+24+29+27+23+19+12+9)
= 406

General formula: $\int_a^b y\,dx = \boldsymbol{w}[\boldsymbol{y_0} + \boldsymbol{y_1} + \boldsymbol{y_2} + \boldsymbol{y_3} + \cdots \boldsymbol{y_{n-1}}]$ where $w = \frac{b-a}{n} = \frac{b-a}{number\ of\ rectangles/strips} = \frac{b-a}{number\ of\ coordinates-1}$

The formula basically says $h[1^{st}\ y\ + 2^{nd}\ y + 3^{rd}\ y + \cdots + (one\ from\ last)y]$

What happens when the widths aren't the same? We can't use the formula as the width cannot be found by $\frac{b-a}{n}$ since you will not be dividing the interval evenly. We do manually it by adding all the individual rectangle areas. Notice how we don't use the x values, we use the widths and the y coordinates!

width width width

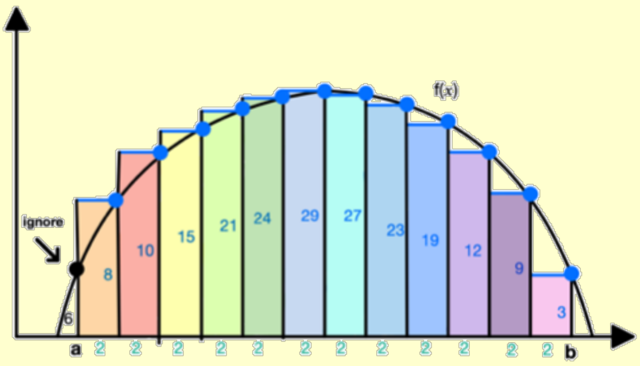
X	a			b
Y	y_0	y_1	y_2	y_3

heights etc

Increasing⟹underestimate and decreasing ⟹overestimate

Right Riemann Sum

Each rectangle's height is determined by evaluating f(x) at the **RIGHT**-hand endpoint of the subinterval the rectangle lives on i.e. the height (y coordinate) of each rectangles is determined by **right** endpoint of each x interval for each rectangle, shown with the blue point



In a table format this looks like:

X	a												b
Y	6	8	10	15	21	24	29	27	23	19	12	9	3

Area = sum of the areas of all rectangles
= (2x8) +(2x10)+(2x15)+(2x21)+(2x24)+(2x29)+(2x27)+(2x23)+(2x19)+(2x12)+(2x9)+(2x3)
= 2 (8+10+15+21+24+29+27+23+19+12+9+3)
= 400

General formula: $\int_a^b y\,dx = \boldsymbol{w}[\boldsymbol{y_1} + \boldsymbol{y_2} + \boldsymbol{y_3} + \cdots \boldsymbol{y_n}]$ where $h = \frac{b-a}{n} = \frac{b-a}{number\ of\ rectangles/strips} = \frac{b-a}{number\ of\ coordinates-1}$

The formula basically says $h[1^{st}\ y\ + 2^{nd}\ y + 3^{rd}\ y + \cdots + (\ last)y]$

What happens when the widths aren't the same? We can't use the formula as the width cannot be found by $\frac{b-a}{n}$ since you will not be dividing the interval evenly. We do manually it by adding all the individual rectangle areas. Notice how we don't use the x values, we use the widths and the y coordinates!

width width width

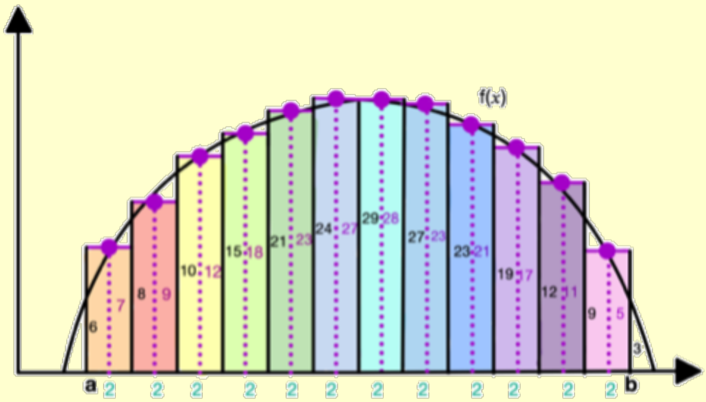
X	a			b
Y	y_0	y_1	y_2	y_3

heights etc

Increasing⟹overestimate and decreasing ⟹underestimate

Midpoint Riemann Sum

Each rectangle's height is determined by evaluating f(x) at the **middle** of the subinterval the rectangle lives on i.e. the height (y coordinate) of each rectangles is determined by **middle** of each x interval for each rectangle, , shown with the **purple point**



In a table format this looks like:

X	a												b
Y	6	7	8	9	10	12	15	18	21	23	24	27	29

Area = sum of the areas of all rectangles
= (2x7) +(2x9)+(2x12)+(2x18)+(2x23)+(2x27)+(2x28)+(2x23)+(2x21)+(2x17)+(2x11)+(2x5)
= 2 (7+9+12+18+23+27+28+23+21+17+11+5)
= 402

General formula:
Notice how we don't use the x values, we use the widths and the y coordinates!

General formula: $\int_a^b y\,dx = \boldsymbol{h}\left[\boldsymbol{y_{\frac{1}{2}} + y_{\frac{3}{2}} + y_{\frac{5}{2}} + \cdots + y_{n-\frac{3}{2}} + y_{n-\frac{1}{2}}}\right]$ where $h = \frac{b-a}{number\ of\ rectangles/strips} = \frac{b-a}{number\ of\ coordinates-1}$

The formula basically says $h[middle\ of\ 1^{st}\ two\ y's\ + middle\ of\ 2^{nd}\ and\ 3^{rd} + middle\ of\ 3^{rd}\ and\ 4^{th} y + \cdots + middle\ of\ one\ from\ ladt\ and\ last\ y]$

What happens when the widths aren't the same? We can't use the formula as the width cannot be found by $\frac{b-a}{n}$ since you will not be dividing the interval evenly. We do manually it by adding all the individual rectangle areas. Notice how we don't use the x values, we use the widths and the y coordinates!

width width width

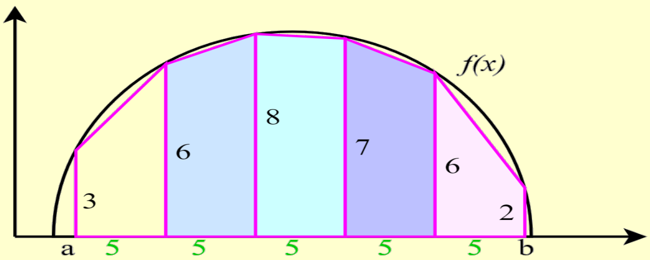
	X	a						b	
heights	Y	y_0	$y_{0.5}$	y_1	$y_{1.5}$	y_2	$y_{2.5}$	y_3	etc

Concave up ⟹overestimate in total and concave down ⟹underestimate in total

The midpoint sum is better than trapezium rule below since the errors balance each other out. It Overestimates AND underestimates. The Mid ordinate is more accurate than trapezium rule below, but trapezium rule is more accurate than left and right sums.

Trapezium Rule

Now drawing trapezia instead of rectangles



5 5 5 5 5

a					b
3	6	8	7	6	2

Area = sum of the areas of all trapezoids
=½ x 5 (3+6) + ½ x 5 (6+8) + ½ x 5 (8+7) + ½ x 5 (7+6) + ½ x 5 (6+2)
= ½ x 5 (3 + 6 + 6 + 8 + 8 + 7 + 7 + 6 + 6 + 2)
= ½ x 5 [3 + 2(6) + 2(8) + 2(7) + 2(6) +2]
=147.5

General formula: $\int_a^b y\,dx = \frac{h}{2}[\boldsymbol{y_0 + 2(y_1 + y_2 + y_3 + y_4 + \cdots) + y_n}]$ where $h = \frac{b-a}{number\ of\ strips} = \frac{b-a}{number\ of\ coordinates-1}$

The formula basically says $\frac{1}{2}h[1st\ y\ + 2(middle\ y's) + lasty]$

What happens when the widths aren't the same? We can't use the formula as the width cannot be found by $\frac{b-a}{n}$ since you will not be dividing the interval evenly. We do manually it by adding all the individual trapezia areas. Notice how we don't use the x values, we use the widths and the y coordinates!

width width width

	X	a			b	
lengths of parallel sides	Y	y_0	y_1	y_2	y_3	etc

concave up⟹overestimate and concave down ⟹underestimate