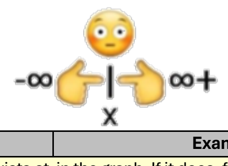
Intuition

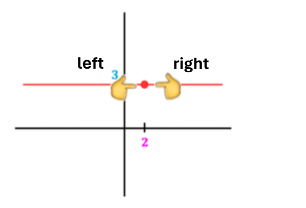
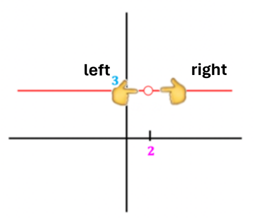
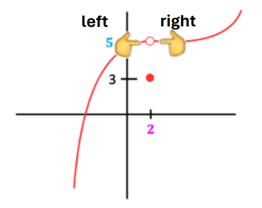
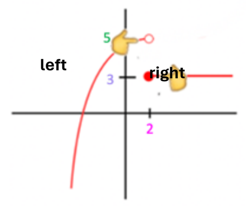
The easiest way to get an intuition about limits is to first look at them graphically.

Informally we say, a limit exists at a point if we can trace THE CURVE (the function) inwards with our fingers from either side of the point (from both the left and right of the value that we are finding the limit at) and tend towards the same value (reach the same height on the axis)

Checking limits be like:

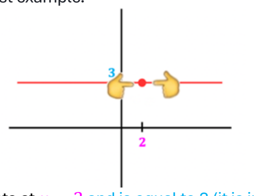


Let's look at the limits of the following functions to demonstrate this idea:

Example 1	Example 2	Example 3	Example 4
Find whether the limit exists at in the graph. If it does, find its value.			
 This limit exists at $x=2$. The value of the function is 3 from both the right and the left and we say the limit is equal to 3	 This limit exists at $x=2$. Notice how we don't know the value of the function at since the circle is not filled in (unlike the example on the left). This doesn't matter. The value of the curve tends to 3 from both the right and the left and we say the limit is equal to 3	 This limit exists at $x=2$. Notice how the value of the curve is 3 (where the closed circle is at), but this does not matter. The value of the curve is tending towards 5 from both the right and the left irrespective of the actual value and we say the limit is equal to 5.	 This limit DOES NOT exist at $x=2$. The value of the limit from the right is equal to 3 and the value of the limit from the left is equal to 5. These values are not the same (not equal), so the limit DOES NOT exist.

Notation

We also have a special **notation** to talk about limits. Let's consider our first example.



We have already seen that the limit of this function in the graph above exists at $x=2$ and is equal to 3 (it is just the y value).

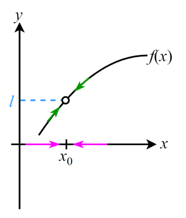
More formally we can write this as

$$\lim_{x\to 2} f(x) = 3$$

In English:

- lim means we're taking a limit of something. A limit is the value that the function **approaches** as we come closer to our desired x value
- The expression $f(x)$ on the right side of the word lim is the expression we're taking the limit of. In our case, that's the function f
- The expression $x\to 2$ that comes below the word lim means that we take the limit of the function f as values of x approaches 2.

More formally, we say, a limit exists at a point x_0 and is equal to l if we can trace the graph of $f(x)$ inwards from either side of the point which has an x-coordinate of x_0 and tend towards the same y value/height which is l



We write this as

$$\lim_{x\to x_0} f(x) = l$$

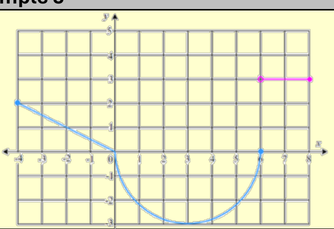
In English this says that as x approaches x_0 (from the left or the right side of x_0 of the function $f(x)$) the function approaches a y value of l.

Type 1: Finding Limits Graphically

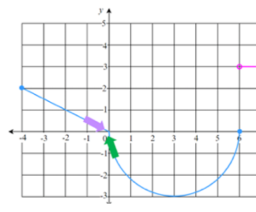
Example 5

Consider the graph to the right which shows the graph of a function $f(x)$

i. Does $\lim_{x\to 0} f(x)$ exist?
ii. Does $\lim_{x\to 6} f(x)$ exist?



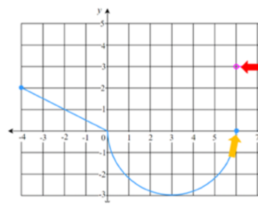
i.



$$\lim_{x\to 0^+} f(x) = 0$$
$$\lim_{x\to 0^-} f(x) = 0$$

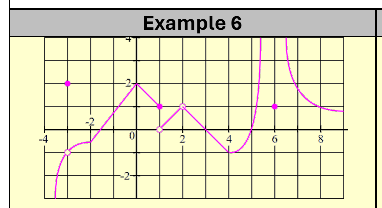
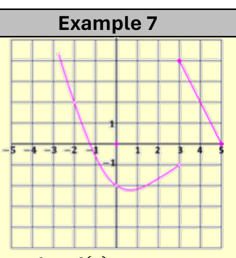
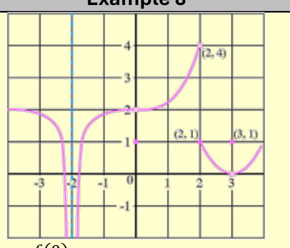
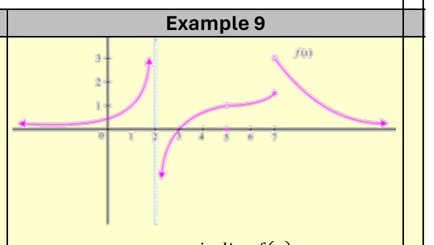
Yes, the limit does exist at $x=0$ since both one-sided limits are the same (they both are equal to zero)

ii.



$$\lim_{x\to 6^+} f(x) = 3$$
$$\lim_{x\to 6^-} f(x) = 0$$

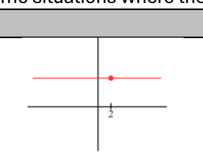
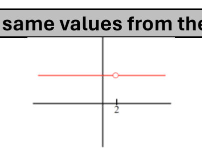
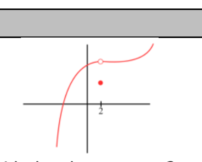
No, the limit does not exist at $x=6$ since the one-sided limits are not equal ($3 \neq 0$)

Example 6	Example 7	Example 8	Example 9
 i. Does $\lim_{x\to -3} f(x)$ exist? If so, state what it is. ii. Does $\lim_{x\to 1} f(x)$ exist? If so, state what it is. iii. Does $\lim_{x\to 2} f(x)$ exist? If so, state what it is. iv. Does $\lim_{x\to 4} f(x)$ exist? If so, state what it is. v. Does $\lim_{x\to 6} f(x)$ exist? If so, state what it is.	 i. $\lim_{x\to -2} f(x)$ ii. $\lim_{x\to 0} f(x)$ iii. $\lim_{x\to 3} f(x)$ iv. $\lim_{x\to 5^-} f(x)$	 i. $f(0)$ ii. $\lim_{x\to -2} f(x)$ iii. $\lim_{x\to 2^-} f(x)$ iv. $\lim_{x\to 3} f(x)$	 i. $\lim_{x\to 5} f(x)$ ii. $\lim_{x\to 2^+} f(x)$ iii. $\lim_{x\to 2^-} f(x)$ iv. $\lim_{x\to 7^+} f(x)$ v. $\lim_{x\to 7^-} f(x)$ vi. $\lim_{x\to 7} f(x)$ vii. $\lim_{x\to\infty} f(x)$
i. $\lim_{x\to -3} f(x) = -1$. Yes limit exists ii. $\lim_{x\to 1} f(x)$ does not exist (tends to different limits of 0 and 1) iii. $\lim_{x\to 2} f(x)=1$ iv. $\lim_{x\to 4} f(x)=-1$ v. $\lim_{x\to 6} f(x)$ does not exist (tends to the same limit but it is infinity)	i. $\lim_{x\to -2} f(x) = 2$ ii. $\lim_{x\to 0} f(x) = -2$ iii. $\lim_{x\to 3} f(x)$ does not exist iv. $\lim_{x\to 5^-} f(x)=0$ The minus just means from the left side	i. $f(0) = 1$ ii. $\lim_{x\to -2} f(x) - \infty$ does not exist ii. $\lim_{x\to 2^-} f(x)=4$ iii. $\lim_{x\to 3} f(x)=0$	i. $\lim_{x\to 5} f(x)=1$ ii. $\lim_{x\to 2^+} f(x)=-\infty$ does not exist iii. $\lim_{x\to 2^-} f(x)=\infty$ iv. $\lim_{x\to 7^+} f(x)=3$ v. $\lim_{x\to 7^-} f(x)=1.5$ vi. $\lim_{x\to 7} f(x)$ does not exist vii. $\lim_{x\to\infty} f(x)=0$

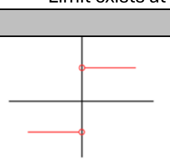
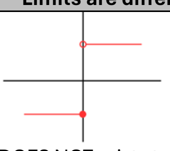
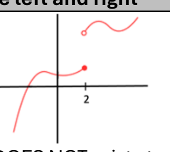
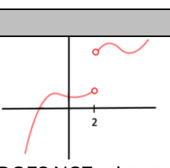
Summary of when a Limit DOES NOT EXIST (DNE) Visually

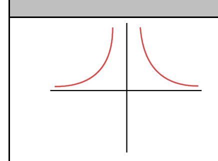
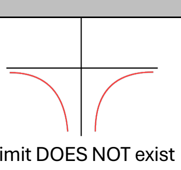
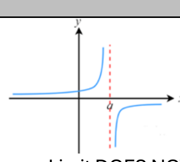
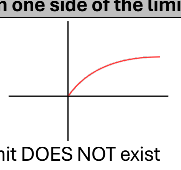
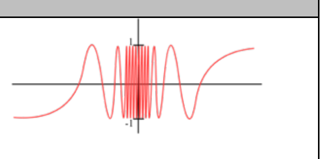
Let's look at some situations where the limit does not exist

Limits are the same values from the left and right

 Limit exists at $x=2$	 Limit exists at $x=2$	 Limit exists at $x=2$

Limits are different values from the left and right

 Limit DOES NOT exist at $x=0$	 Limit DOES NOT exist at $x=0$	 Limit DOES NOT exist at $x=2$	 Limit DOES NOT exist at $x=2$

Limits tend to infinity			Function not defined on one side of the limit	Limit oscillates on either eaither
 Limit DOES NOT exist at $x=0$. The limit diverges to ∞ on both sides of zero which is not a number, hence we say the limit DNE.	 Limit DOES NOT exist at $x=0$. The limit diverges to $-\infty$ on both sides of zero . This is not a number, hence we say the limit DNE.	 Limit DOES NOT exist at $x=a$. The limit from the right is $-\infty$ and the limit from the left is ∞. These are unequal and also not numbers.	 Limit DOES NOT exist at $x=0$. The limit from the right is 0, but the function isn't defined for values to the left of $x=0$ meaning there is no limit from the left. Thus, DNE.	 Limit DOES NOT exist at $x=0$. This is because of oscillatory behaviour. The graph of the function oscillates infinitely up and down as approaches 0. $f(x)$ oscillates between −1 and 1, hence there is no single value for the limit to exist.

Type 2: Finding Algebraic Limits

Immediate Substitution

Tending to a number

For the easiest types of questions, we can simply substitute the numbers into the function and we are done. Let's see how this work with a few basic examples.

Example 6	Example 7	Example 8	Example 9
Find $\lim_{x\to 3} 2x$	Find $\lim_{x\to 2}(x^2+5x)$	Find $\lim_{x\to 0}\frac{x^2+4}{x-2}$	Find $\lim_{x\to 2}(2x-1)^4$
$\lim_{x\to 3} 2x$ This tells us to replace x with 3 in $2x$. Substituting gives $2(3) = 6$	$\lim_{x\to 2}(x^2+5x)$ This tells us to replace x with 2 in x^2+5x. Substituting gives $2^2+5(2) = 14$	$\lim_{x\to 0}\frac{x^2+4}{x-2}$ This tells us to replace x with 0 in $\frac{x^2+4}{x-2}$. Substituting gives $\frac{0^2+4}{0-2} = -2$	$\lim_{x\to 2}(2x-1)^4$ This tells us to replace x with 2 in $(2x-1)^4$. Substituting gives $(2(2)-1)^4 = 81$

For all the above examples we say the limit exists and whatever number we get is the value of the limit.

Tending to infinity and zero

However, we don't always get a non-zero number or 'nice' numbers. Sometimes we get zero and undefined answers. Let's look at a few examples

Example 10	Example 11	Example 12	Example 13
Find $\lim_{x\to 0}\frac{x}{5}$	Find $\lim_{x\to 0}\frac{10}{x}$	Find $\lim_{x\to\infty}\frac{x}{2}$	Find $\lim_{x\to\infty}\frac{3}{x}$
Substitute $x=0$: $\lim_{x\to 0}\frac{x}{5} = \frac{0}{5}$. Zero divided by a non-zero number is always zero. $=0$. We say the limit **exists** and is equal to **zero.**	Substitute $x=0$: $\lim_{x\to 0}\frac{10}{x} = \frac{10}{0}$. We cannot divide by zero. We say the limit **does not exist**	Substitute $x=\infty$: $\lim_{x\to\infty}\frac{x}{2} = \frac{\infty}{2}$. A very very large number over a much smaller negligible number in comparison will remain a very very big number $=\infty$. We say the limit **DNE**. This is because is not a number.	Substitute $x=\infty$: $\lim_{x\to\infty}\frac{3}{x} = \frac{3}{\infty}$. If we divide by a very very large number we practically get 0. *Think about it. The more slices you cut a cake into the smaller the slices become. If we kept cutting the cake size would get smaller and smaller until eventually we would barely get any cake.* $=0$. We say the limit **exists** and is equal to **zero.**

Summary of when Limit DOES & DOES NOT EXIST (DNE) algebraically

$\frac{\text{non-zero number}}{0} = DNE$. We say the limit **doesn't exist**. See example 11	$\frac{\pm\infty}{\text{non-infinite number}} = \pm\infty$. We say the limit **doesn't exist**. See example 12	If sub in and get a real number, we say the limit **DOES exist** and is equal to the number. See examples 6-9	$\frac{0}{\text{any non-zero number}} = 0$. We say the limit **DOES exist** and is equal to **zero**. See example 10	$\frac{\text{any non-infinite number}}{\pm\infty} = 0$. We say the limit **DOES exist** and is equal to **zero**. See example 13

You should also know the following

$\lim_{x\to\infty}\frac{1}{x} = 0$ $\quad\lim_{x\to-\infty}\frac{1}{x} = 0$ $\quad\lim_{x\to\infty} e^x = \infty$ $\quad\lim_{x\to-\infty} e^x = 0$ $\quad\lim_{x\to\infty}\ln x = \infty$ $\quad\lim_{x\to\infty}\log x = \infty$ $\quad\lim_{x\to\infty} e^{-x} = 0$ $\quad\lim_{x\to-\infty} e^{-x} = \infty$

Unfortunately, we can also get worse than examples 6-13. We can substitute in and get what is known as an indeterminate form.

Indeterminate Form – Do 'something' first

There are seven indeterminate forms that we can get when substituting in that we need to be aware about. They are:

$$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad \infty-\infty \qquad 0(\infty) \qquad 0^0 \qquad 1^\infty \qquad \infty^0$$

These indeterminate forms **don't tell us whether** the limit exists or does not exist. Hence, they are "indeterminate". Instead, it tells us we need to try some other approaches.

Factorise First to Cancel Terms

This method comes in handy when we deal with $\frac{0}{0}$ **form**, also the numerator and denominator are **polynomials.** These two things should signal the use of factorising:

Step 1: Factorise the numerator and/or denominator
Step 2: Write the original expression using the factored terms
Step 3: Cancel like terms from the numerator and denominator
Step 4: Direct substitute to get the final answer

Example 14	Example 15	Example 16
Find $\lim_{x\to 4}\frac{4-x}{x^2-16}$	Find $\lim_{x\to -1}\frac{2x^2-x-3}{x+1}$	Find $\lim_{x\to 2}\frac{x^3-8}{x^2+x-6}$
First, we check direct substitution: $\frac{4-4}{4^2-16} = \frac{0}{0}$. This is $\frac{0}{0}$ and has polynomials. Step 1: $x^2-16 = (x-4)(x+4)$, $4-x = -(x-4)$. Step 2: $\lim_{x\to 4}\frac{-(x-4)}{(x-4)(x+4)}$. Step 3: $\lim_{x\to 4}\frac{-\cancel{(x-4)}}{\cancel{(x-4)}(x+4)}$. Step 4: $\frac{-1}{4+4} = -\frac{1}{8}$	First, we check direct substitution: $\frac{2(-1)^2-(-1)-3}{-1+1} = \frac{0}{0}$. This is $\frac{0}{0}$ and has polynomials. Step 1: $2x^2-x-3 = (x+1)(2x-3)$. Step 2: $\lim_{x\to -1}\frac{(x+1)(2x-3)}{(x+1)}$. Step 3: $\lim_{x\to -1}\frac{\cancel{(x+1)}(2x-3)}{\cancel{(x+1)}}$. Step 4: $2(-1)-3 = -5$	First, we check direct substitution: $\frac{2^3-8}{2^2+2-6} = \frac{0}{0}$. This is $\frac{0}{0}$ and has polynomials. Step 1: $x^3-8 = (x-2)(x^2+2x+4)$, $x^2+x-6 = (x+3)(x-2)$. Step 2: $\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x+3)(x-2)}$. Step 3: $\lim_{x\to 2}\frac{\cancel{(x-2)}(x^2+2x+4)}{(x+3)\cancel{(x-2)}}$. Step 4: $\frac{2^2+2(2)+4}{2+3} = \frac{12}{5}$

Rationalise First To Get Rid Of The Roots

This method comes in handy when we deal with $\frac{0}{0}$ **form** when the numerator and denominator have **radicals/square roots.**

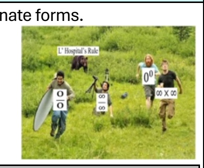
Step 1: Multiply the numerator and denominator by the conjugate
Step 2: Expand the brackets (using $(a+b)(a-b) = a^2-b^2$ formula) and simplify the terms
Step 3: Direct substitute to get the final answer

Example 17	Example 18	Example 19
Find $\lim_{x\to -1}\frac{x+1}{\sqrt{x+5}-2}$	Find $\lim_{y\to 0}\frac{\sqrt{5+y}-\sqrt{5}}{y}$	Find $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$
First, we check substitution: $\frac{-1+1}{\sqrt{-1+5}-2} = \frac{0}{0}$. This is $\frac{0}{0}$ and has radicals hence can rationalise. Step 1: $\frac{x+1}{\sqrt{x+5}-2}\times\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2}$. Step 2: $\frac{(x+1)(\sqrt{x+5}+2)}{(\sqrt{x+5})^2-2^2} = \frac{\cancel{(x+1)}(\sqrt{x+5}+2)}{\cancel{x+1}}$. Step 3: $\sqrt{-1+5}+2 = 4$	First, we check substitution: $\frac{\sqrt{5+0}-\sqrt{5}}{0} = \frac{0}{0}$. This is $\frac{0}{0}$ and has radicals hence can rationalise. Step 1: $\frac{\sqrt{5+y}-\sqrt{5}}{y}\times\frac{\sqrt{5+y}+\sqrt{5}}{\sqrt{5+y}+\sqrt{5}}$. Step 2: $\frac{(\sqrt{5+y})^2-(\sqrt{5})^2}{y(\sqrt{5+y}+\sqrt{5})} = \frac{\cancel{y}}{\cancel{y}(\sqrt{5+y}+\sqrt{5})}$. Step 3: $\frac{1}{\sqrt{5+0}+\sqrt{5}} = \frac{1}{2\sqrt{5}}$	First, we check substitution: $\frac{\sqrt{1+0}-\sqrt{1-0}}{0} = \frac{0}{0}$. This is $\frac{0}{0}$ and has radicals hence can rationalise. Step 1: $\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$. Step 2: $\frac{(\sqrt{1+x})^2-(\sqrt{1-x})^2}{x(\sqrt{1+x}+\sqrt{1-x})} = \frac{2\cancel{x}}{\cancel{x}(\sqrt{1+x}+\sqrt{1-x})}$. Step 3: $\frac{2}{\sqrt{1+0}+\sqrt{1-0}} = 1$

Use L'Hôpital's Rule First

This is one of the most important and powerful rules when solving limits algebraically. It is useful for solving multiple kinds of indeterminate forms.

The rule itself can **only be applied for $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$ forms.** The process is as follows:



Step 1: Ensure direct substitution into the limit gives one of the two forms: $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
Step 2: Differentiate the numerator, differentiate the denominator
Step 3: Rewrite the limit as $\frac{\text{differentiated numerator}}{\text{differentiated denominator}}$ and solve this as a new question (you may have to use L'Hôpital's again!)

Example 20	Example 21	Example 22
Find $\lim_{x\to 0}\frac{\sin x}{x}$	Find $\lim_{x\to\infty}\frac{\ln x}{x}$	Find $\lim_{x\to\infty}\frac{e^x}{x^2}$
Step 1: $\frac{\sin 0}{0} = \frac{0}{0}$. This is $\frac{0}{0}$ hence we apply L'Hôpital's. Step 2: $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(x) = 1$. Step 3: $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = \cos(0) = 1$. **Hence, $\lim_{x\to 0}\frac{\sin x}{x} = 1$**	Step 1: $\frac{\ln\infty}{\infty} = \frac{\infty}{\infty}$. This is $\frac{\infty}{\infty}$ hence we apply L'Hôpital's. Step 2: $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $\frac{d}{dx}(x) = 1$. Step 3: $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$. **Hence, $\lim_{x\to\infty}\frac{\ln x}{x} = 0$**	Step 1: $\frac{e^\infty}{\infty^2} = \frac{\infty}{\infty}$. This is $\frac{\infty}{\infty}$ hence we apply L'Hôpital's. Step 2 & 3: $\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x}$. Notice, directly substituting here gives $\frac{e^\infty}{2(\infty)} = \frac{\infty}{\infty}$. This is $\frac{\infty}{\infty}$ hence we apply L'Hôpital's AGAIN. Repeated L'Hôpital's makes it powerful in solving complex limits: $\lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = e^\infty = \infty$. Hence, limit DNE

Three Common Mistakes

Mistake 1: Using L'Hopitals Without An Indeterminate form	Mistake 2: Using The quotient Rule	Mistake 3: Solving when limit DNE (see the one- sided limits section on the page below to be able to understand this)
Find $\lim_{x\to 0}\frac{\cos x}{x}$	Find $\lim_{x\to\infty}\frac{\ln x}{x}$	Find $\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}$
$\frac{\cos 0}{0} = \frac{1}{0}$. Since this is indeterminate, we can rewrite the expression with L'Hôpital's giving us $\lim_{x\to 0}\frac{\cos x}{x} = \lim_{x\to 0}\frac{-\sin x}{1}$. Here, direct substitution gives $\frac{-\sin 0}{1} = 0$	$\frac{\ln\infty}{\infty} = \frac{\infty}{\infty}$. This is $\frac{\infty}{\infty}$ hence we apply L'Hôpital's $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2}$, $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1-\ln x}{x^2}$. And so on…	$\frac{\sin 0}{\sqrt{0}} = \frac{0}{0}$. Since this is $\frac{0}{0}$, we can apply L'Hôpital's $\lim_{x\to 0}\frac{\sin x}{\sqrt{x}} = \lim_{x\to 0}\frac{\cos x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0} 2\sqrt{x}\cos x$. Here, direct substitution gives $2\sqrt{0}\cos 0 = 0$
Note that is not the correct form for applying L'Hôpital's. The expression must be $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to apply this rule	Note that we differentiate the numerator and denominator SEPARATELY and then write it as a fraction. Do not apply the quotient rule onto the whole fraction.	The mistake is much **harder to spot** here. Recall that a limit only exists if both one-sided limits exist. Here, $\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}}$ exists, however $\lim_{x\to 0^-}\frac{\sin x}{\sqrt{x}}$ doesn't because $\sqrt{x}$ isn't defined for $x<0$.

Type 3: Finding Trigonometric Limits

Immediate Substitution

Example 23	Example 24	Example 25	Example 26
Find $\lim_{x\to 1}\cos\frac{\pi x}{3}$	Find $\lim_{x\to 0}\frac{x^2-2}{\cos x}$	Find $\lim_{x\to\frac{\pi}{4}}\sin^3 x\sec^4 x$	Find $\lim_{x\to\pi}(3\sin x-2x)$
$\lim_{x\to\pi}(3\sin x-2x)$. This tells us to replace x with π in $(3\sin x-2x)$. Substituting gives $(3\sin\pi-2\pi) = -2\pi$	$\lim_{x\to\pi}(3\sin x-2x)$. This tells us to replace x with π in $(3\sin x-2x)$. Substituting gives $(3\sin\pi-2\pi) = -2\pi$	$\lim_{x\to\pi}(3\sin x-2x)$. This tells us to replace x with π in $(3\sin x-2x)$. Substituting gives $(3\sin\pi-2\pi) = -2\pi$	$\lim_{x\to\pi}(3\sin x-2x)$. This tells us to replace x with π in $(3\sin x-2x)$. Substituting gives $(3\sin\pi-2\pi) = -2\pi$

Indeterminate Form – Do 'something' first

Use Trig Identities First

When limits have trigonometric functions, we often use our trigonometric identities to simplify the question and progress forward. If you ever get confused which identity to use pick the one which takes you to the basic trig identities sin or cos

Here's a quick revision.

Reciprocal Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$
Pythagorean Identities	$\sin^2\theta+\cos^2\theta = 1$	$1+\tan^2\theta = \sec^2\theta$	$1+\cot^2\theta = \operatorname{cosec}^2\theta$	
Addition and Subtraction Identities	$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta\pm\cos\alpha\sin\beta$	$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta\mp\sin\alpha\sin\beta$		
Double Angle Identities	$\sin(2\theta) = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta-\sin^2\theta = 1-2\sin^2\theta = 2\cos^2\theta-1$	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	

Example 27	Example 28	Example 29
Find $\lim_{x\to 0}\frac{\sin x}{\sin 2x}$	Find $\lim_{x\to-\frac{\pi}{4}}\frac{1+\sqrt{2}\sin x}{\cos 2x}$	Find $\lim_{x\to\frac{\pi}{2}}\frac{\cot^2 x}{1-\sin x}$
$\frac{\sin(0)}{\sin 2(0)} = \frac{0}{0}$. This is $\frac{0}{0}$ hence we could apply L'Hôpital's (this also gives us the same answer) but we can utilise a double angle identity to replace the $\sin 2x$: $\lim_{x\to 0}\frac{\sin x}{\sin 2x} = \lim_{x\to 0}\frac{\cancel{\sin x}}{2\cancel{\sin x}\cos x} = \frac{1}{2\cos 0} = \frac{1}{2}$	$\frac{1+\sqrt{2}\sin\left(-\frac{\pi}{4}\right)}{\cos\left(2\left(-\frac{\pi}{4}\right)\right)} = \frac{1+\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)}{\cos\left(-\frac{\pi}{2}\right)} = \frac{0}{0}$. This is $\frac{0}{0}$ hence we could apply L'Hôpital's (this also gives us the same answer) but we can utilise a double angle identity to replace the $\cos 2x$: $\lim_{x\to-\frac{\pi}{4}}\frac{1+\sqrt{2}\sin x}{\cos 2x} = \lim_{x\to-\frac{\pi}{4}}\frac{1+\sqrt{2}\sin x}{1-2\sin^2 x}$ where we use difference of squares identity $\lim_{x\to-\frac{\pi}{4}}\frac{\cancel{1+\sqrt{2}\sin x}}{\cancel{(1+\sqrt{2}\sin x)}(1-\sqrt{2}\sin x)} = \frac{1}{\left(1-\sqrt{2}\sin\left(-\frac{\pi}{4}\right)\right)} = \frac{1}{2}$	$\frac{\cot^2\frac{\pi}{2}}{1-\sin\frac{\pi}{2}} = \frac{0}{0}$. This is $\frac{0}{0}$ hence we could apply L'Hôpital's (this also gives us the same answer) but we can utilise a reciprocal identity to replace the cot x: $\lim_{x\to\frac{\pi}{2}}\frac{\cot^2 x}{1-\sin x} = \lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{(\sin^2 x)(1-\sin x)}$. Now, we see the denominator is fully in terms of $\sin x$, so we can use Pythagorean identity on numerator: $\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{(\sin^2 x)(1-\sin x)} = \lim_{x\to\frac{\pi}{2}}\frac{1-\sin^2 x}{(\sin^2 x)(1-\sin x)}$ where we use difference of squares identity $\lim_{x\to\frac{\pi}{2}}\frac{(1+\sin x)\cancel{(1-\sin x)}}{(\sin^2 x)\cancel{(1-\sin x)}} = \frac{1+\sin\frac{\pi}{2}}{\sin^2\frac{\pi}{2}} = 2$

Use L'Hôpital's Rule First

Example 30 (L'Hopitals)	Example 31 (L'Hopitals)	Example 32 (L'Hopitals)
Find $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$	Find $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}$	Find $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\cos 2x}$
Do not always look for identities just because there is a trigonometric function, also use the previously learned concepts. Here, direct substitution gives $\frac{\sin a(0)}{\sin b(0)} = \frac{0}{0}$. This indetermine form of $\frac{0}{0}$ means we can use the L'Hôpital's rule as follows $\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{a\cos ax}{b\cos bx} = \frac{a\cos 0}{b\cos 0} = \frac{a}{b}$	This question feels intimidating at first, but keep in mind everything we know. Direct substitution gives us $\frac{1-\tan\frac{\pi}{4}}{1-\sqrt{2}\sin\frac{\pi}{4}} = \frac{1-1}{1-1} = \frac{0}{0}$. This indeterminate form of $\frac{0}{0}$ indicates using L'Hôpital's rule. $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{x\to\frac{\pi}{4}}\frac{-\sec^2 x}{-\sqrt{2}\cos x} = \lim_{x\to\frac{\pi}{4}}\frac{1}{\sqrt{2}\cos^3 x}$. Here, we can do direct substitution to get $\frac{1}{\sqrt{2}\cos^3\left(\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^3} = 2$	Direct substitution gives us $\frac{1-\tan\frac{\pi}{4}}{\cos\left(2\left(\frac{\pi}{4}\right)\right)} = \frac{0}{0}$. This indeterminate form of $\frac{0}{0}$ indicates using L'Hôpital's rule. But, let's try some trigonometric identities to test our skills: $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\cos 2x} = \lim_{x\to\frac{\pi}{4}}\frac{1-\frac{\sin x}{\cos x}}{\cos^2 x-\sin^2 x} = \lim_{x\to\frac{\pi}{4}}\frac{\cancel{\cos x-\sin x}}{\cos x(\cos x+\sin x)\cancel{(\cos x-\sin x)}} = \frac{1}{\cos\frac{\pi}{4}\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)} = 1$

Use Small Angle Approximations First (can only when tending to zero)

When a trigonometric limit has $x\to 0$, we can use a set of useful "approximations" for the trigonometric functions. These approximations can simplify the problems and speed up working. Here are the ones you must know:

$$\sin x \approx \tan x \approx x$$
$$\cos x \approx 1-\frac{x^2}{2}$$

Let us look at a familiar question to test this approximation:

$$\lim_{x\to 0}\frac{\sin x}{x}$$

While we would usually use L'Hôpital's rule here, let us try our approximation

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1$$

Similarly,

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1$$

These questions, however, seem equally fast by the small angle approximation or by L'Hôpital's rule. Instead, let us see the following question where this approximation actually saves us a lot of time:

Example 33

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

Way 1: Small Angle Approximation	Way 2: L'Hôpital's rule
$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-\left(1-\frac{x^2}{2}\right)}{x^2} = \lim_{x\to 0}\frac{\frac{x^2}{2}}{x^2} = \frac{1}{2}$. **Thus, our answer is $\frac{1}{2}$**	First, we do direct substitution $\frac{1-\cos 0}{0^2} = \frac{0}{0}$. Hence, the $\frac{0}{0}$ form allows us to use L'Hôpital's rule giving $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x}$. Here, direct substitution again gives $\frac{0}{0}$ hence, $\lim_{x\to 0}\frac{\sin x}{2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{\cos 0}{2} = \frac{1}{2}$. **Thus, we have our answer $\frac{1}{2}$**

Type 4: Finding Infinite Limits With Rational Functions

This is for when we have $x\to\infty$ and our function is a fraction with polynomials in the numerator and denominator

$$\lim_{x\to\infty}\frac{2x+3}{4x^2-5}$$

Bottom-heavy (denominator has the highest power)	Even powers (a tie with highest power of numerator & denominator)	Top-heavy (numerator has the highest power)
Example 34	**Example 35**	**Example 36**
$\lim_{x\to\infty}\frac{2x+3}{4x^2-5}$	$\lim_{x\to\infty}\frac{5x^2+5x-5}{6x^2-2x+5}$	$\lim_{x\to\infty}\frac{10x^2+x}{4x-1}$
$\lim_{x\to\infty}\frac{2x+3}{4x^2-5}$. Highest power in denominator is a 2 due to x^2 term, so let's divide all terms by this. $\lim_{x\to\infty}\frac{\frac{2x}{x^2}+\frac{3}{x^2}}{\frac{4x^2}{x^2}-\frac{5}{x^2}} = \lim_{x\to\infty}\frac{\frac{2}{x}+\frac{3}{x^2}}{4-\frac{5}{x^2}}$. Substitute and get $\frac{0+0}{4-0} = \frac{0}{4} = 0$. Shortcut method: Here the highest power in the denominator is greater than that of the numerator hence bottom-heavy (the bottom becomes very large) and the limit is always going to equal 0	$\lim_{x\to\infty}\frac{5x^2+5x-5}{6x^2-2x+5}$. Highest power in denominator is a 2 due to x^2 term, so let's divide all terms by this. $\lim_{x\to\infty}\frac{\frac{5x^2}{x^2}+\frac{5x}{x^2}-\frac{5}{x^2}}{\frac{6x^2}{x^2}-\frac{2x}{x^2}+\frac{5}{x^2}} = \lim_{x\to\infty}\frac{5+\frac{5}{x}-\frac{5}{x^2}}{6-\frac{2}{x}+\frac{5}{x^2}}$. Substitute and get $\frac{5+0-0}{6-0+0} = \frac{5}{6}$. Shortcut method: Here the ratio of the highest power terms which are x terms is $\frac{5}{6}$. Hence limit $=\frac{5}{6}$	$\lim_{x\to\infty}\frac{10x^2+x}{4x-1}$. Highest power in denominator is a 1 due to term, so let's divide all terms by this. $\lim_{x\to\infty}\frac{\frac{10x^2}{x}+\frac{x}{x}}{\frac{4x}{x}-\frac{1}{x}} = \lim_{x\to\infty}\frac{10x+1}{4-\frac{1}{x}}$. Substitute and get $\frac{\infty}{4-0} = \infty$. Shortcut method: Here the highest power in the numerator is greater than that of the denominator hence top-heavy and hence there is no limit (infinite)

Careful when you have a root in the denominator. If $x\to\infty$, divide inside the root by x^2 and outside by x

$$\lim_{x\to\infty}\frac{2x}{\sqrt{4x^2+1}} = \lim_{x\to\infty}\frac{\frac{2x}{x}}{\frac{1}{x}\sqrt{4x^2+1}} = \lim_{x\to\infty}\frac{\frac{2x}{x}}{\sqrt{\frac{4x^2}{x^2}+\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{2}{\sqrt{4+\frac{1}{x^2}}}$$

Here, we substitute to get

$$=\frac{2}{\sqrt{4+0}} = \frac{2}{2} = 1$$

Harder Limit Examples

Example 37

Find $\lim_{x\to 2}\frac{\sqrt{6-x}-2}{3-\sqrt{11-x}}$

This is a clear case for rationalising. However, we see radicals in the numerator and denominator both, hence we multiply and divide by two conjugates at the same time, giving us

$$\frac{\sqrt{6-x}-2}{3-\sqrt{11-x}}\times\frac{3+\sqrt{11-x}}{3+\sqrt{11-x}}\times\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}$$

Hence, we use the difference of two squares identity to get

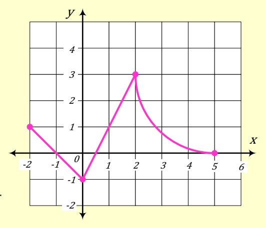
$$\frac{\left(\left(\sqrt{6-x}\right)^2-2^2\right)\left(3+\sqrt{11-x}\right)}{\left(3^2-\left(\sqrt{11-x}\right)^2\right)\left(\sqrt{6-x}+2\right)}=\frac{(6-x-4)\left(3+\sqrt{11-x}\right)}{(9-(11-x))\left(\sqrt{6-x}+2\right)}=\frac{-\cancel{(x-2)}\left(3+\sqrt{11-x}\right)}{\cancel{(x-2)}\left(\sqrt{6-x}+2\right)}$$

Hence, giving

$$\lim_{x\to 2}\frac{\sqrt{6-x}-2}{3-\sqrt{11-x}}=\lim_{x\to 2}\frac{-\left(3+\sqrt{11-x}\right)}{\sqrt{6-x}+2}=\frac{-\left(3+\sqrt{11-2}\right)}{\sqrt{6-2}+2}=-\mathbf{\frac{3}{2}}$$

Example 38

t (hours)	0	0.3	1	2.8	4
$v_p(t)$ (meters per hour)	0	55	-29	55	48



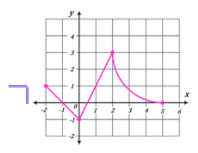
Graph of f

Find $\lim_{t\to 1}\frac{e^t-3f(t)}{v_p(t)-\cos(\pi t)}$

This question involves using algebra, table, and graphs all at once.
Let us start by direct substitution as

$$\lim_{t\to 1}\frac{e^t-3f(t)}{v_p(t)-\cos(\pi t)}=\frac{e^1-3f(1)}{v_p(1)-\cos(\pi)}$$

Hence, we can find

$e^1=e$	$\cos(\pi)=-1$
$3f(1)=3(1)=3$ 	$v_p(1)=-29$

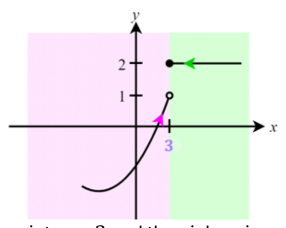
t (hours)	0	0.3	1	2.8	4
$v_p(t)$ (meters per hour)	0	55	-29	55	48

Thus,

$$\lim_{t\to 1}\frac{e^t-3f(t)}{v_p(t)-\cos(\pi t)}=\frac{e-3}{-29-(-1)}=\mathbf{\frac{3-e}{28}}$$

Type 5: One Sided Limits Versus Two Sided Limits

Tracing the curve from EITHER SIDE (one side at a time i.e. from the left or from the right) is called **one-sided** limits.
Let's look at an example to best understand this



The green region represents the part of the graph to the right of the point $x=3$ and the pink region represents the part of the graph to the left of this point. We use **superscripts** (powers which are either plus or minus) to indicate these regions:

- $\lim_{x\to 3^-} f(x)$ is called a left limit or left-hand limit. The **negative superscript** signifies from the **left side** (values **less than** 3). This means we trace/approach the graph from the **left side.**

$$\lim_{x\to 3^-} f(x)=1$$

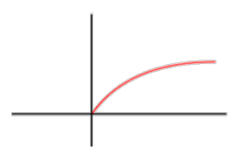
- $\lim_{x\to 3^+} f(x)$ is called a right limit or right-hand limit. The **positive superscript** signifies this from the **right side** (values **greater than** 3). This means we trace/approach the graph from the **right side.**

$$\lim_{x\to 3^+} f(x)=2$$

Note: our familiar $\lim_{x\to 3} f(x)$ is called a two-sided limit. It is basically the limit tested from **both** directions

If both one-sided limits are **not equal** then the two-sided limit does not exist.
If both one-sided limits are **equal**, then the two-sided limit, $\lim_{x\to 3} f(x)$ exists. If this two-sided limit exists, we call it l and write the two-sided limit as $\lim_{x\to 3} f(x)=l$.
These one-sided limits are not equal since Hence the two-sided limit $\lim_{x\to 3} f(x)$ is undefined and we say the limit does not exist

Note: Sometimes a function is only defined on one side, and in that case, a one-sided limit is the best we can do.
For example, we can only talk about the right sided limit at $x=0$ for the function below. $\lim_{x\to 0^+} f(x)=0$
but $\lim_{x\to 0^-} f(x)$ does not exist since there is no function on that side.



Example 39 (with piecewise function)

$f(x)=\begin{cases}3x-7, & x\le 2\\ x^2-5, & x>2\end{cases}$

Determine if $\lim_{x\to 2} f(x)$ exists

Here we have two functions. Let's colour code.

$$f(x)=\begin{cases}3x-7, & x\le 2\\ x^2-5, & x>2\end{cases}$$

f(x) behaves as $3x-7$ for values of $x\le 2$ and as x^2-5 for $x>2$

We want to find the limit as $x\to 2$, but the function behaves differently on either side, so we have to find both one-sided limits.

$$\lim_{x\to 2^+} f(x)=(2)^2-5=-1$$
$$\lim_{x\to 2^-} f(x)=3(2)-7=-1$$

Both one-sided limits equal -1, so the limit, $\lim_{x\to 2} f(x)$ exists

Example 40 (with modulus)

Find $\lim_{x\to 2}\frac{|x-2|}{x-2}$

We need to have knowledge of the modulus functions in order to be able to write this as a piecewise function
We can write this as a piecewise function:

$$f(x)=\begin{cases}\frac{x-2}{x-2}=1, & x\ge 2\\ \frac{-(x-2)}{x-2}=-1, & x<2\end{cases}$$

Simplifying gives

$$f(x)=\begin{cases}1, & x\ge 2\\ -1, & x<2\end{cases}$$

We want to find the limit as $x\to 2$, but the function behaves differently on either side, so we have to find both one-sided limits.

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} 1=1$$
$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} -1=-1$$

One limit is -1 and the other is 1
so the limit, $\lim_{x\to 2}\frac{|x-2|}{x-2}$ doesn't exist

Example 41

Find $\lim_{x\to 1^+}\frac{1-x^2}{|1-x|}$

This involves modulus, which means we will create a piecewise function as

$$\frac{1-x^2}{|1-x|}=\frac{(1-x)(1+x)}{|1-x|}$$

$$=\begin{cases}\frac{\cancel{(1-x)}(1+x)}{-\cancel{(1-x)}}=-1-x, & x>1\\ \frac{\cancel{(1-x)}(1+x)}{\cancel{(1-x)}}=1+x, & x\le 1\end{cases}$$

Hence, we have

Example 42 (with superscript notation)

Find $\lim_{x\to -1^-}\frac{2x+1}{3x-1}$ and $\lim_{x\to -1^+}\frac{2x+1}{3x-1}$

$$\lim_{x\to -1^-}\frac{2x+1}{3x-1}=\frac{2(-1)+1}{3(-1)-1}=\frac{1}{4}$$

$$\lim_{x\to -1^+}\frac{2x+1}{3x-1}=\frac{2(-1)+1}{3(-1)-1}=\frac{1}{4}$$

Notice how approaching from either side means the same thing for the value, hence

Example 42 (with superscript notation)

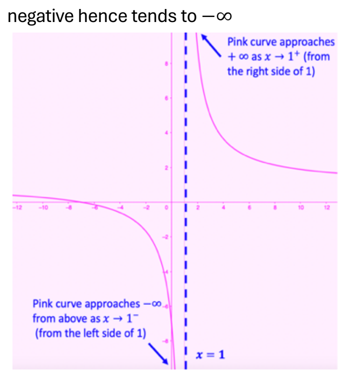
Find $\lim_{x\to 1^-}\frac{x+7}{x-1}$ and $\lim_{x\to 1^+}\frac{x+7}{x-1}$

This is harder since we get $\frac{0}{0}$ for both when we substitute, but we need to know whether our answer is ∞ or $-\infty$.

$$\lim_{x\to 1^+}\frac{x+7}{x-1}=\infty \text{ and } \lim_{x\to 1^-}\frac{x+7}{x-1}=-\infty$$

If you're struggling to see why 1^+ gives ∞ and 1^- gives $-\infty$, consider the following:

- $\lim_{x\to 1^+}$ means plug in a value just greater than 1, say 1.00001 making the denominator positive hence tends to $+\infty$
- $\lim_{x\to 1^-}$ means plug in a value just less than one, say 0.999999 making the denominator negative hence tends to $-\infty$



Example 43

Find $\lim_{x\to\infty}\frac{x+8}{x-3}$ and $\lim_{x\to -\infty}\frac{x+8}{x-3}$

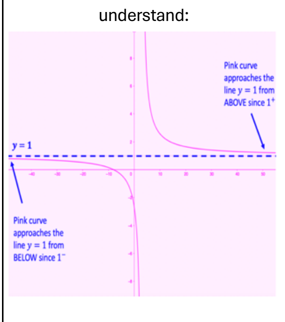
$$\lim_{x\to\infty}\frac{x+8}{x-3}=1^+$$

$$\lim_{x\to -\infty}\frac{x+8}{x-3}=1^-$$

We write 1^+ and 1^- which both mean 1. The sign just tells us the direction that we approach the line $y=1$ from which is the asymptote of the graph

1^+ means we approach the line $y=1$ from above as x tends to infinity

1^- means we approach the line $y=1$ from below as x tends to negative infinity
Let's look at this graphically to understand:



Example 44

Find $\lim_{x\to\infty}\frac{x+3}{(5-x)^2}$ and $\lim_{x\to -\infty}\frac{x+3}{(5-x)^2}$

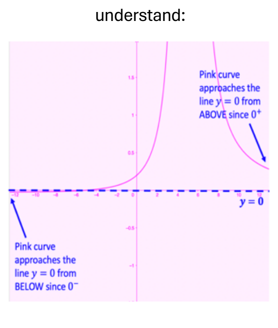
$$\lim_{x\to\infty}\frac{x+3}{(5-x)^2}=0^+$$

$$\lim_{x\to -\infty}\frac{x+3}{(5-x)^2}=0^-$$

We write 0^+ and 0^- which both mean 0. The sign just tells us the direction that we approach the line $y=0$ from which is the asymptote of the graph

0^+ means we approach the line $y=0$ from above as x tends to infinity

0^- means we approach the line $y=0$ from below as x tends to negative infinity
Let's look at this graphically to understand:



Type 6: Squeeze Theorem

This is one of the rarest methods you will come across when solving limits. It involves *SQUEEZING* the desired limit between two other functions.

What does squeezing mean?
It means finding two functions who are always bigger and always smaller than our desired limit's function.

Let us take a simple example to demonstrate:

Example 45

$$\lim_{x\to\infty}\frac{\sin x}{x}$$

Direct substitution gives

$$\frac{\sin\infty}{\infty}$$

Hmm, we do not know what $\sin\infty$ is because the graph of sine keeps going up and down, hence there's no ONE value that $\sin\infty$ could equal. So, let us try squeezing the function.

First, we set up the range of the function which is giving us problems.

$$-1\le\sin\infty\le 1$$

How do we know this inequality? This inequality comes from the range of $\sin x$ which is only $[-1,1]$.
Hence, we can get the function in our limit by dividing x on all sides giving us

$$-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$$

This is how we set up the Squeeze Theorem. Next, we just apply the limit on all sides giving

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right)\le\lim_{x\to\infty}\frac{\sin x}{x}\le\lim_{x\to\infty}\frac{1}{x}$$

Hence, solving for the values we know, we get

$$0\le\lim_{x\to\infty}\frac{\sin x}{x}\le 0$$

Now, for the main aspect of the Squeeze Theorem.
If the limit is greater than or equal to 0 and the limit is smaller than or equal to 0, then the limit must be equal to zero. Hence,

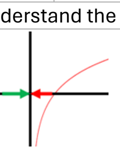
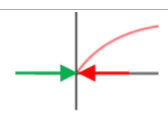
$$\lim_{x\to\infty}\frac{\sin x}{x}=0$$

Common Limits and Functions That You Must Understand

You should already be familiar with

$\lim_{x\to\infty}\frac{1}{x}=0$	$\lim_{x\to-\infty}\frac{1}{x}=0$	$\lim_{x\to\infty} e^x=\infty$	$\lim_{x\to-\infty} e^x=0$
$\lim_{x\to\infty}\ln x=\infty$	$\lim_{x\to\infty}\log x=\infty$	$\lim_{x\to\infty} e^{-x}=0$	$\lim_{x\to-\infty} e^{-x}=\infty$

Now you should also understand the limits for harder questions

$\lim_{x\to 0^+}\ln x=-\infty$ $\lim_{x\to 0^-}\ln x=DNE$  Hence, $\lim_{x\to 0}\ln x=DNE$	$\lim_{x\to 0^+}\sqrt{x}=0$ $\lim_{x\to 0^-}\sqrt{x}=DNE$  Hence, $\lim_{x\to 0}\sqrt{x}=DNE$
$\lim_{x\to\infty}\log_r x=-\infty\ (0<r<1)$	$\lim_{x\to 0^+}\log_r x=\infty\ (0<r<1)$
$\lim_{x\to\infty}\log_r x=\infty\ (r>1)$	$\lim_{x\to 0^+}\log_r x=-\infty\ (r>1)$
$\lim_{x\to\infty} r^x=\infty\ (r>1)$	$\lim_{x\to\infty} r^x=0\ (\lvert r\rvert<1)$

Hardest Limit Examples

Harder L'Hôpital's Rule

Other harder forms where it is not so obvious that we can use L'Hopital's rule (i.e. non-fractions) can be transformed to apply this rule.

On the first page we only saw the use of L'Hôpital's for the indeterminate forms stated in the rule, i.e. $\frac{0}{0}$ and $\frac{\infty}{\infty}$ but this rule has a lot more power helping us solve $0(\pm\infty)$
If the limits looks like a product where one term is 0 and the other is infinity, we can turn it into a quotient. In other words if we have

$$\lim_{x\to a} f(x)g(x)\ \text{ where } f(a)=0,\ g(a)=\pm\infty$$

L'Hôpital's rule can only be applied on quotients, so we convert this product into a quotient question as follows:

$$\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}\text{ where } f(a)=0,\ \frac{1}{g(a)}=\frac{1}{\pm\infty}=0$$

This allows us to apply L'Hôpital's rule easily now as the indeterminate form of $\frac{0}{0}$ is reached. NOTE that we could have put $f(x)$ in the denominator as well giving us

$$\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}\text{ where } g(a)=0,\ \frac{1}{f(a)}=\frac{1}{0}=\pm\infty$$

Deciding if we want $f(x)$ or $g(x)$ in denominator will come with practice, however we will also see why one choice is clearly easier or sometimes the only useful one.

Example 46

Find $\lim_{x\to 0^+} x\ln x$

First, we try direct substitution:

$$(0^+)\ln(0^+)=0(-\infty)$$

Hence, we must convert to quotient to apply L'Hôpital's. Notice, putting x in the denominator makes $\frac{1}{x}$ which we know is easy to differentiate. Hence,

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\text{ where substitution gives }\frac{-\infty}{\infty}$$

Thus, we have the form $\frac{-\infty}{\infty}$ and we can apply L'Hôpital's

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=-x=-0^+$$

This gives us

$$\lim_{x\to 0^+} x\ln x=0$$

Example 47

Find $\lim_{x\to-\infty} xe^x$

First, we try direct substitution:

$$(-\infty)(e^{-\infty})=(-\infty)(0)$$

Hence, we must convert to quotient to apply L'Hôpital's. Notice, both x and e^x are reasonable candidates to put down. Let us try first

$$\lim_{x\to-\infty} xe^x=\lim_{x\to-\infty}\frac{e^x}{\frac{1}{x}}\text{ where substitution gives }\frac{0}{0}$$

Thus, we have the form $\frac{0}{0}$ and we can apply L'Hôpital's

$$\lim_{x\to-\infty}\frac{e^x}{\frac{1}{x}}=\lim_{x\to-\infty}\frac{e^x}{-\frac{1}{x^2}}$$

Notice, we once again have $\frac{0}{0}$ and we can apply L'Hôpital's

$$\lim_{x\to-\infty}\frac{e^x}{-\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{e^x}{\frac{2}{x^3}}$$

Notice, we once again have $\frac{0}{0}$. Clearly, our choice of x for putting in the denominator was incorrect. So, let us try e^x instead.

$$\lim_{x\to-\infty} xe^x=\lim_{x\to-\infty}\frac{x}{\frac{1}{e^x}}=\lim_{x\to-\infty}\frac{x}{e^{-x}}\text{ where substitution gives }\frac{-\infty}{\infty}$$

Thus, we have the form $\frac{-\infty}{\infty}$ and we can apply L'Hôpital's

$$\lim_{x\to-\infty}\frac{x}{e^{-x}}=\lim_{x\to-\infty}\frac{1}{-e^{-x}}=\lim_{x\to-\infty}-e^x=-e^{-\infty}=0$$

This gives us

$$\mathbf{\lim_{x\to-\infty} xe^x=0}$$

Example 48

Find $\lim_{x\to\infty} x^3e^{-x}$

First, we try direct substitution:

$$\infty^3e^{-\infty}=\infty(0)$$

Hence, we must convert to quotient to apply L'Hôpital's. Notice, the question automatically tells us to take e^{-x} to the denominator due to its negative exponent. Hence,

$$\lim_{x\to\infty} x^3e^{-x}=\lim_{x\to\infty}\frac{x^3}{e^x}$$

Thus, we have $\frac{\infty}{\infty}$ and can apply L'Hôpital's multiple times (since each time we get the indeterminate form $\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\frac{x^3}{e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}=\lim_{x\to\infty}\frac{6}{e^x}=\frac{6}{e^\infty}=0$$

This gives us

$$\mathbf{\lim_{x\to\infty} x^3e^{-x}=0}$$

Example 49

Find $\lim_{x\to\infty} x^ne^{-x}$ where $n\in\mathbb{N}$

We can write this limit as a fraction to get

$$\lim_{x\to\infty}\frac{x^n}{e^x}$$

where plugging in gives $\frac{\infty}{\infty}$ allowing to apply L'Hôpital's rule

$$\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}$$

Where plugging in again gives $\frac{\infty}{\infty}$ allowing to apply L'Hôpital's rule

$$\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}=\lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x}$$

Notice, we keep getting $\frac{\infty}{\infty}$ and thus we continuously apply L'Hôpital's rule as

$$\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}=\lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x}=\cdots=\lim_{x\to\infty}\frac{n(n-1)\dots(2)x}{e^x}=\lim_{x\to\infty}\frac{n(n-1)\dots(2)(1)}{e^x}$$

This just gives us:

$$\lim_{x\to\infty}\frac{n!}{e^x}$$

Here, we have a constant on the numerator and ∞ in the denominator hence we get

$$\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{n!}{e^x}=\mathbf{0}$$

Harder Squeeze Theorem

Example 50 (harder squeeze theorem)

Find $\lim_{x\to 0^-} x^3\cos\frac{2}{x}$

Direct substitution gives

$$0^3\cos\frac{2}{0}$$

Now, we cannot find $\cos\frac{2}{0}$, hence we don't know what $\cos\frac{2}{0}$ evaluates to. We can utilise the squeeze theorem here as

$$-1\le\cos\frac{2}{x}\le 1$$

Now, we multiply by x^3. Here it is important to note one thing from the question. Our limit is tending to 0^-, which means $x<0$. Hence, we know that x^3 must be negative as well. This is important because when multiplying by a negative value, the inequality must flip the sign. Hence, we get

$$-x^3\ge x^3\cos\frac{2}{x}\ge x^3$$

Hence, applying the limit gives

$$\lim_{x\to 0^-}(-x^3)\ge\lim_{x\to 0^-}x^3\cos\frac{2}{x}\ge\lim_{x\to 0^-}x^3$$
$$0\ge\lim_{x\to 0^-}x^3\cos\frac{2}{x}\ge 0$$

Thus, by the squeeze theorem, we have that

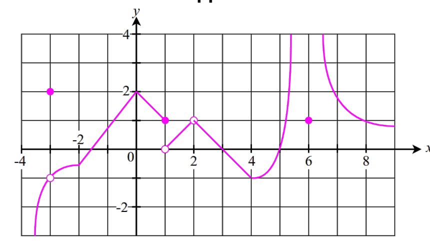
$$\lim_{x\to 0^-}x^3\cos\frac{2}{x}=0$$

Summaries

Showing limit exists or DNE

Graphically

Ensure same value approached from both sides



Does $\lim_{x\to -3} f(x)$ exist?
$\lim_{x\to -3^+} f(x)=-1$ and $\lim_{x\to -3^-} f(x)=-1$
$\lim_{x\to -3^+} f(x)=\lim_{x\to -3^-} f(x)$
$\therefore\lim_{x\to -3} f(x)$ exists

Does $\lim_{x\to 2} f(x)$ exist?
$\lim_{x\to 2^+} f(x)=1$ and $\lim_{x\to 2^-} f(x)=1$
$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^-} f(x)$
$\therefore\lim_{x\to 2} f(x)$ exists

Does $\lim_{x\to 1} f(x)$ exist?
$\lim_{x\to 1^+} f(x)=0$ and $\lim_{x\to 1^-} f(x)=1$
$\lim_{x\to 1^+} f(x)\ne\lim_{x\to 1^-} f(x)$
$\therefore\lim_{x\to 1} f(x)$ does not exist

Does $\lim_{x\to 6} f(x)$ exist?
$\lim_{x\to 6^+} f(x)=\infty$ and $\lim_{x\to 6^-} f(x)=\infty$
$\lim_{x\to 6^+} f(x)=\lim_{x\to 6^-} f(x)$
Since ∞ is not a number, $\lim_{x\to 6} f(x)$ does not exist

Algebraically

Solve to check non-infinite answer is found

Example where a limit exists:

$$\lim_{x\to 3} 2x$$

This tells us to replace x with 3 in the expression $2x$

Substituting gives

$$2(3)$$
$$=6$$

So, we get a number which means that the limit exists

Example where a limit doesn't exist:

$$\lim_{x\to 0}\frac{2}{x}$$

Substituting gives $\frac{2}{0}$ which is undefined because we cannot divide by zero and therefore the limit DNE

Piecewise

Check both one-sided limits are equal

Example where a limit exists:

$$f(x)=\begin{cases}3x-7, & x\le 2\\ x^2-5, & x>2\end{cases}$$

Determine $\lim_{x\to 2} f(x)$ if it exists

Here we have two functions.

$$f(x)=\begin{cases}3x-7, & x\le 2\\ x^2-5, & x>2\end{cases}$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(3x-7)=3(2)-7=-1$$
$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x^2-5)=(2)^2-5=-1$$

Both limits equal -1 so the limit exists

Example where a limit doesn't exist:

$$\lim_{x\to 2}\frac{|x-2|}{x-2}$$

We can write this as a piecewise function:

$$f(x)=\begin{cases}\frac{x-2}{x-2}=1, & x\ge 2\\ \frac{-(x-2)}{x-2}=-1, & x<2\end{cases}$$

Simplifying gives

$$f(x)=\begin{cases}1, & x\ge 2\\ -1, & x<2\end{cases}$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} 1=1$$
$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} -1=-1$$

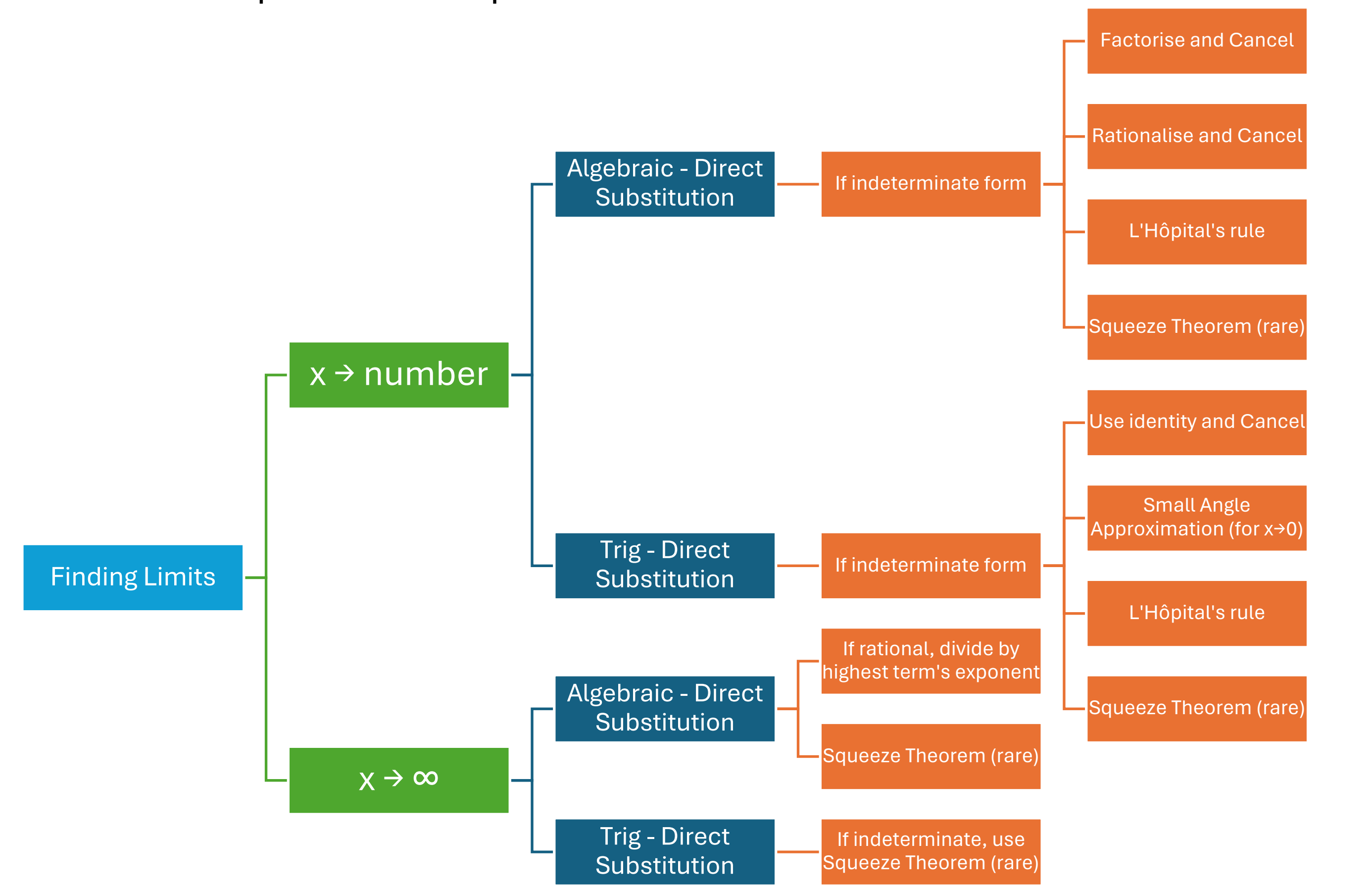
One limit is -1 and the other is 1 so the limit DNE

Calculating Limits

Do I have a fraction and can I factorise and cancel? Your factorising needs to be good (see factorising cheat sheet if not)	Do I have a fraction with a square root in the numerator or denominator? Rationalise!	Do I have a fraction that gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and can I differentiate the numerator and denominator?	Do I have trig functions? Can I use the identities: • $\sin^2 x+\cos^2 x=1$ • $1+\tan^2 x=\sec^2 x$ • $1+\cot^2 x=\mathrm{cosec}^2 x$ • $\sin 2x=2\sin x\cos x$ • $\cos 2x=\cos^2 x-\sin^2 x$ • $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\sin\beta\cos\alpha$ • $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\pm\sin\beta\sin\alpha$	Do I have trig and is my limit tending to zero? $\sin x\approx x$ $\cos x\approx 1-\frac{x^2}{2}$ $\tan x\approx x$	Is $x\to\pm\infty$ and both numerator and denominator are polynomials? Do I get $\frac{\pm\infty}{\pm\infty}$ when I substitute?	Have the 6 methods to the left failed? Use squeeze theorem! This is for $x\to 0$ and $x\to\infty$ (This type usually occurs for trig, factorials or powers of x)
Factorise and cancel **Example:** $\lim_{x\to 2}\frac{x^2-4x+4}{x-2}$ substitution yields $\frac{0}{0}$ Let's factorise first: $\lim_{x\to 2}\frac{\cancel{(x-2)}(x-2)}{\cancel{x-2}}$ $=\lim_{x\to 2} x-2$ Now we can substitute $2-2$ $=0$	**Rationalise** the numerator or denominator and cancel **Example:** $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$ substitution yields $\frac{0}{0}$ Let's rationalise the numerator Consider the function $\frac{\sqrt{x+5}-3}{x-4}\times\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$ $\lim_{x\to 4}\frac{x+5-9}{(x-4)(\sqrt{x+5}+3)}$ $=\lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x+5}+3)}$ Now we can cancel $\lim_{x\to 4}\frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x+5}+3)}$ $=\frac{1}{\sqrt{4+5}+3}$ $=\frac{1}{6}$	Apply **L' Hopital's Rule** It is important to know that you **can only** apply L' Hopital's Rule as long as you have an indeterminate form: or . We just differentiate the numerator and denominator. **Example:** $\lim_{x\to -3}\frac{x^3+9x^2+27x+27}{x^3+8x^2+21x+18}$ Substitution yields $\frac{0}{0}$ Hence, L'Hopital's rule (differentiate top/bottom) $\lim_{x\to -3}\frac{3x^2+18x+27}{3x^2+16x+21}$ Substitution still yields $\frac{0}{0}$ Use L'Hopital's again $\lim_{x\to -3}\frac{6x+18}{6x+16}$ Substitute $=\frac{6(-3)+18}{6(-3)+16}=\frac{0}{-2}$ 	If trig we use a **trig identity** and maybe simplify after **Example:** $\lim_{x\to\pi}\frac{\tan^2 x}{1+\sec x}$ Substitution yields $\frac{0}{0}$ Let's use a trig identity to get $\lim_{x\to\pi}\frac{\sec^2 x-1}{1+\sec x}$ Use difference of two squares $\lim_{x\to\pi}\frac{\cancel{(\sec x+1)}(\sec x-1)}{\cancel{1+\sec x}}$ Hence, $\lim_{x\to\pi}(\sec x-1)$ $=\sec\pi-1$ $=-1-1$ $=-2$	Use **small angle approximations** We **can only do this** if trig and $x\to 0$. It is important that you realise this! As soon as a limit tends to zero for trig, you should check whether small angle approximations help! **Example:** $\lim_{x\to 0}\frac{\sin x}{x}$ Substitution yields $\frac{0}{0}$ small angle approximation gives: $\lim_{x\to 0}\frac{x}{x}$ $=\lim_{x\to 0} 1$ $=1$ when $\sin\theta=\theta$: 	We can apply a shortcut for these types. If the denominator wins, then the limit will equal 0 If the numerator wins, the limit will equal $\pm\infty$ hence it DNE. If there is a tie, the limit will be the ratio of terms with those coefficients of terms with highest power.	**Squeeze theorem!** This basically says if we can find a smaller and bigger function that have the same limit, then the function "squeezed between them" must also have the same limit. **Example 1:** $\lim_{x\to 0} x\cos\frac{1}{x}$ Substitution gives $0\times\cos\frac{1}{0}$ Let's use squeeze theorem Compare to $-\lvert x\rvert$ and $\lvert x\rvert$ $\lim_{x\to 0}-\lvert x\rvert=0$ $\lim_{x\to 0}\lvert x\rvert=0$ Since $-1\le\cos\left(\frac{1}{x}\right)\le 1$, Hence $-\lvert x\rvert\le x\cos\left(\frac{1}{x}\right)\le\lvert x\rvert$ Thus, the answer is as $\lim_{x\to 0} x\cos\frac{1}{x}=0$ 

Mind Map

We can think of our entire process as a mind map



A More Detailed Mind Map