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Differentiation Rules – Short Method

Type	Rule	Examples	Comments
"Easy Powers" aka power rule	$x^n \Rightarrow nx^{n-1}$ "bring the power down to the front and subtract one from the power"	$y = 3x^4 + 4x^2 - 5x + 6$ $\frac{dy}{dx} = 12x^3 + 8x - 5$ **Don't forget: terms with an x go to their coefficient and constants go to zero**	This is the first type you learn and should find this easy by now

We also deal with harder composite function as $f(g(x))$. In other words, we now have to deal with a function within a function, or a function of a function whereby there is an inside function and an outside function.

$\cos(\underbrace{1-x^2}_{\text{inner}})$ outer $(\underbrace{1-x^2}_{\text{inner}})^2$ outer $e^{\overbrace{1-x^2}^{\text{inner}}}$ outer $\ln(\underbrace{1-x^2}_{\text{inner}})$ outer

We must apply what is known as "chain rule". There is a long and short way to apply chain rule for composite functions. It is MUCH quicker to apply the rules as laid out below (see the cheat sheet differentiation rules - shorter and longer methods if you want both methods. The longer method involves a substitution). We can remember the rule by:

"derivative of the outer function times the derivative of the inner function" **OR** *"differentiate the mother and leave the baby inside times the differentiate the baby"*

"Harder Powers"

Rule: $(f(x))^n \Rightarrow n(f(x))^{n-1}f'(x)$

"bring the power down to the front and subtract one from the power (keep what is inside the bracket the same)" and then multiply by the derivative of what is inside the bracket

Examples:

$y = 4(3x^3 - 4x)^3$

$\frac{dy}{dx} = 4(3)(3x^3 - 4x)^2(9x^2 - 4)$

$= 12(9x^2 - 4)(3x^3 - 4x)^2$

Comments: This is similar to year 1 power rule except we now have a function inside the bracket and have to apply chain rule by differentiating the inner function

Exponentials

Type 1: With a base e

$e^{f(x)} \Rightarrow e^{f(x)}f'(x)$

"copy the exponential" and multiply by the derivative of the power

Type 2: With a base other than e

$a^{f(x)} \Rightarrow a^{f(x)}f'(x)\ln a$

"copy the exponential" and multiply by the derivative of the power and multiply by $\ln a$

Why is this second result true? You need to know the proof of this!

Proof 1: Use implicit differentiation	Proof 2: Write $a^{f(x)} = e^{\ln a^{f(x)}}$
$y = a^{f(x)}$ Log both sides $\ln y = \ln a^{f(x)}$ $\ln y = f(x)\ln a \Leftrightarrow \ln y = \ln a\, f(x)$ $\frac{1}{y}\frac{dy}{dx} = \ln a\, f'(x)$ $\frac{dy}{dx} = y\ \ln a\, f'(x)$ Put back y gives $\frac{dy}{dx} = a^{f(x)}\ln a\, f'(x)$ $= a^{f(x)}f'(x)\ln a$	We can use log rules to bring the power down $e^{f(x)\ln a} = e^{\ln a\, f(x)}$ Now this just becomes like type 1 differentiation with a base of e $\Rightarrow e^{\ln a\, f(x)}\ \ln a\, f'(x)$ $= \ln a\ f'(x)e^{\ln a\, f(x)}$ $= \ln a\ f'(x)e^{f(x)\ln a}$ Use a log rule again $= \ln a\ f'(x)e^{\ln a^{f(x)}}$ $= \ln a\ f'(x)a^{f(x)}$

Examples:

$y = 4e^{x^2}$

$\frac{dy}{dx} = 4e^{x^2}(2x) = 8xe^{x^2}$

$y = 3(4^{x^2})$

$\frac{dy}{dx} = 3(4^{x^2})(2x)\ln 4$

$= 6(\ln 4)x4^{x^2}$

Comments: Notice how the $\ln a$ appears for the second example, which doesn't happen when we differentiate with a base of e

Natural Logarithmic Function

Rule: $\ln(f(x)) \Rightarrow \frac{f'(x)}{f(x)}$

This goes to a fraction: $\frac{\text{derivative of argument}}{\text{copy of argument}}$

Note: If given a log instead we turn into $\ln$ first

$\log_a f(x) = \frac{\ln f(x)}{\ln a} = \frac{1}{\ln a}\ln f(x) \Rightarrow \frac{1}{\ln a}\frac{f'(x)}{f(x)}$

Examples:

$y = \ln(x^2 + 2x)$

$\frac{dy}{dx} = \frac{2x+2}{x^2+2x} = \frac{2x+2}{x^2+2x}$

Comments: Don't fall into the trap of always writing $\frac{1}{x}$. It's only $\frac{1}{x}$ for $\ln x$ because $\frac{\text{derivative}}{\text{function}}$ happens to be $\frac{1}{x}$

Trig

Rule:

$\sin f(x) \Rightarrow f'(x)\cos f(x)$

$\cos f(x) \Rightarrow -f'(x)\sin f(x)$

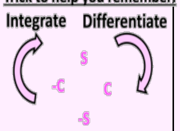
$\tan f(x) \Rightarrow f'(x)\sec^2 f(x)$

$\sec f(x) \Rightarrow f'(x)\sec f(x)\tan f(x)$

$cosec\, f(x) \Rightarrow -f'(x)cosec\, f(x)\cot f(x)$

$\cot f(x) \Rightarrow -f'(x)cosec^2 f(x)$

Trick to help you remember:



"change the trig function to what it is meant to go to (keep the angle the same) and multiply by the derivative of the angle

Examples:

$y = \sin(4x^2)$

$\frac{dy}{dx} = 8x\cos(4x^2)$

$= 8x\cos(4x^2)$

$y = 2\cos^3(x^2)$

First let's write this in power form

$y = 2(\cos(x^2))^3$

Now this is just a harder power type, but be careful when differentiating the $f(x)$ inside the bracket as you have to use the rule for trig also

$\frac{dy}{dx} = 2(3)(\cos(x^2))^2(-2x\sin(x^2))$

$= -12x\sin(x^2)\cos^2(x^2)$

Comments: Watch out for the second example

Trig with powers!

Write this as $y = 2(\cos(x^2))^3$ and then these are just a harder power type and trig in one question

Inverse Trig

Rule:

$\sin^{-1} f(x) = \arcsin f(x) \Rightarrow \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

$\cos^{-1} f(x) = \arccos f(x) \Rightarrow -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$

$\tan^{-1} f(x) = arctan f(x) \Rightarrow \frac{f'(x)}{1+(f(x))^2}$

$\sec^{-1} f(x) \Rightarrow \frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$

$cosec^{-1} f(x) \Rightarrow -\frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$

$\cot^{-1} f(x) \Rightarrow -\frac{f'(x)}{1+(f(x))^2}$

Examples:

$y = \sin^{-1} 4x^2$

Way 1: Use the formula on the left

$\frac{dy}{dx} = \frac{8x}{\sqrt{1-(4x^2)^2}} = \frac{8x}{\sqrt{1-16x^4}}$

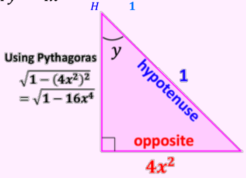
Way 2: Use implicit differentiation (if you already know it)

Get rid of inverse notation: $\sin y = 4x^2$

Differentiate implicitly: $\cos y\frac{dy}{dx} = 8x \Leftrightarrow \frac{dy}{dx} = \frac{8x}{\cos y}$

Need to eliminate the letter y as want everything in terms of x. To do this we build a triangle with $sin = \frac{opp}{hyp}$ and use Pythagoras to find 3rd side. From there we can find what $\cos y$ is in term of x

$\sin y = 4x^2 = \frac{O}{H} = \frac{4x^2}{1}$



Using the triangle and SOHCAHTOA We can find $\cos y$ in terms of x:

$\cos y = \frac{\sqrt{1-16x^4}}{1} = \sqrt{1-16x^4}$

$\therefore \frac{dy}{dx} = \frac{8x}{\cos y}$ can now be written in terms of x as $\frac{8x}{\sqrt{1-16x^4}}$

Note: Instead of building a triangle to eliminate y we could have used the pythagorean identitiy $\sin^2 y + \cos^2 y = 1$. We're given $\sin y = 4x^2$ in the question, so can sub this into $\sin^2 y + \cos^2 y = 1$

$(4x^2)^2 + \cos^2 y = 1 \Rightarrow \cos^2 y = 1 - 16x^4 \Rightarrow \cos y = \sqrt{1-16x^4}$

$\frac{dy}{dx} = \frac{8x}{\cos y} = \frac{8x}{\sqrt{1-16x^4}}$

Comments: Way 2 is longer as you can see. It is much quicker to just memorise the rules

Product & Quotient Rule

(a combination of 2 or more of the 6 types above multiplied together or divided be each other)

Product rule:	Quotient rule:
$y = uv \Rightarrow \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ (differentiate 1st function)(copy 2nd function)+(copy 1st function)(differentiate 2nd function) Ultimately you're just differentiating one function at a time and adding them together	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ This basically says $\frac{(copy\ denominator)(differentiate\ numerator) - (copy\ numerator)(differentiate\ denominator)}{(denominator)^2}$

Basic Examples

Harder Powers:

$y = (3x-2)^5$. Find $\frac{dy}{dx}$

$$y = (3x-2)^5$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the harder powers type. The rule for harder powers is:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

$$\frac{dy}{dx} = 5(3x-2)^4(3)$$

Let's re-order and simplify since we can multiply in any order.

$$\frac{dy}{dx} = 5(3)(3x-2)^4$$

$$\boxed{\frac{dy}{dx} = 15(3x-2)^4}$$

$3(5+x^2)^{\frac{3}{2}}$. Find $\frac{dy}{dx}$.

$$y = 3(5+x^2)^{\frac{3}{2}}$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the harder powers type. The rule for harder powers is:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

https://www.khanacademy.org/math/trigonometry/trig-equations-and-identities/solving-sinusoidal-models/e/inverse-trig-word-problems?modal=1$\frac{dy}{dx} = 3\left(\frac{3}{2}\right)(5+x^2)^{\frac{1}{2}}(2x)$

Let's re-order and simplify since we can multiply in any order

$$\frac{dy}{dx} = 3\left(\frac{3}{2}\right)(2)\,(x)(5+x^2)^{\frac{1}{2}}$$

$$\boxed{\frac{dy}{dx} = 9x(5+x^2)^{\frac{1}{2}}}$$

$f(x) = \frac{5}{\sqrt{2-4x}}$. Find $\frac{dy}{dx}$.

Firstly, we can bring the power up using indices rules and then it is just the harder powers type of differentiation

$$f(x) = 5(2-4x)^{-\frac{1}{2}}$$

$$f'(x) = 5\left(-\frac{1}{2}\right)(2-4x)^{-\frac{3}{2}}(-4)$$

$$\boxed{f'(x) = 10(2-4x)^{-\frac{3}{2}}}$$

Note: A lot of students will try and use quotient rule for this. It is not necessary to use quotient rule here, but you can and it would work!

$$f(x) = \frac{5}{\sqrt{2-4x}}$$

We use quotient rule when we have division of one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) (here we have a constant and a harder power type and a constant is not considered one of the necessary types, that is why is it not necessary to use quotient rule). Also note that we can even always avoid quotient rule when we do have division of one of the 6 differentiation types, by bringing the denominator up and always just using product rule, but this can require more simplification at the end. Therefore, I would suggest using Quotient rule when it is necessary.

Exponentials: Base e

$y = e^{4x}$. Find $\frac{dy}{dx}$.

$$y = e^{4x}$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the exponentials type. The rule for exponentials with base *e* is:

"**copy the entire exponential and then multiply by the derivative of the power**"

$$\frac{dy}{dx} = e^{4x}(4)$$

$$\boxed{\frac{dy}{dx} = 4e^{4x}}$$

$y = 5e^{5x}$. Find $\frac{dy}{dx}$.

$$y = 5e^{5x}$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the exponentials type. The rule for exponentials with base *e* is:

"**copy the entire exponential and then multiply by the derivative of the power**"

$$\frac{dy}{dx} = 5e^{5x}(5)$$

$$\boxed{\frac{dy}{dx} = 25e^{5x}}$$

Exponentials: Base other than e

$y = 2^{4x}$. Find $\frac{dy}{dx}$.

$y = 2^{4x}$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the exponentials type. The rule for exponentials (with a base other than e) is:

"**copy the entire exponential and then multiply by the derivative of the power and also by ln of the base**"

Notice the extra purple part when we have an exponential which doesn't have a base of e.

$$\frac{dy}{dx} = 2^{4x}(4)\ln 2$$

$$\boxed{\frac{dy}{dx} = (4\ln 2)(2^{4x})}$$

$y = 5(3^{2x})$. Find $\frac{dy}{dx}$.

$y = 5(3^{2x})$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the exponentials type. The rule for exponentials (with a base other than e) is:

"**copy the entire exponential and then multiply by the derivative of the power and also by ln of the base** "

Notice the extra purple part when we have an exponential which doesn't have a base of e.

Don't worry about the 5 at the front, that is just hanging around at the front.

$$\frac{dy}{dx} = 5(3^{2x})(2)\ln 3$$

simplify

$$\boxed{\frac{dy}{dx} = (10\ln 3)(3^{2x})}$$

Natural Logarithms:

$y = \ln(3x+2)$. Find $\frac{dy}{dx}$.

$y = \ln(3x+2)$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the logs type. The rule for logs is:

"**This turns into to a fraction which looks like $\frac{\textit{derivative of argument}}{\textit{copy of argument}}$. Notice how the ln disappears**"

$$\boxed{\frac{dy}{dx} = \frac{3}{3x+2}}$$

$y = 3\ln(x^2 + 3x + 5)$. Find $\frac{dy}{dx}$.

$$y = 3\ln(x^2 + 3x + 5)$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the logs type. The rule for logs is:

"This turns into to a fraction which looks like $\frac{\textit{derivative of argument}}{\textit{copy of argument}}$. Notice how the ln disappears"

Don't worry about the 3 at the front, that is just hanging around at the front.

$$\frac{dy}{dx} = 3\left(\frac{2x+3}{x^2+3x+5}\right)$$

simplify

$$\boxed{\frac{dy}{dx} = \frac{6x+9}{x^2+3x+5}}$$

Trig:

$y = \cos 3x$. Find $\frac{dy}{dx}$.

$$y = \cos(3x)$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the trig type. The rule for trig is:

"Change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle"

Our 6 trig functions that we have to remember are:

$\boldsymbol{sin \Rightarrow cos}$
$\boldsymbol{cos \Rightarrow -sin}$
$\boldsymbol{tan \Rightarrow sec^2}$
$\boldsymbol{sec \Rightarrow sec\ tan}$
$\boldsymbol{cosec \Rightarrow -cosec\ cot}$
$\boldsymbol{cot \Rightarrow -cosec^2}$

Here we have the 2nd one: $\boldsymbol{cos \Rightarrow -sin}$

$$\frac{dy}{dx} = -\sin 3x\,(3)$$

simplify

$$\boxed{\frac{dy}{dx} = -3\sin 3x}$$

Harder Examples

$$y = \sin(4x^2). \text{ Find } \frac{dy}{dx}.$$

$$y = sin\,(4x^2)$$

We have 6 types of differentiation (easy powers, harder powers, exponentials, logs, trig and inverse trig). This is the trig type. The rule for trig is:

"Change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle"

Our 6 trig functions that we have to remember are:
$\boldsymbol{sin \Rightarrow cos}$
$\boldsymbol{cos \Rightarrow -sin}$
$\boldsymbol{tan \Rightarrow sec^2}$
$\boldsymbol{sec \Rightarrow sec\,tan}$
$\boldsymbol{cosec \Rightarrow -cosec\,cot}$
$\boldsymbol{cot \Rightarrow -cosec^2}$

Here we have the 1st one: $\boldsymbol{sin \Rightarrow cos}$

$$\frac{dy}{dx} = \cos(4x^2)\,(8x)$$

simplify

$$\boxed{\frac{dy}{dx} = 8x\cos(4x^2)}$$

$$y = e^{\cos x}$$

Here we have a mix of 2 types (exponential and trig)

Recall the rule for exponentials:

"copy the entire exponential and then multiply by the derivative of the power"

Recall the rule for trig:

"Change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle"
Our 6 trig functions that we have to remember are:
$sin \Rightarrow cos$
$\boldsymbol{cos \Rightarrow -sin}$ **(we have this one in this example)**
$tan \Rightarrow sec^2$
$sec \Rightarrow sec\,tan$
$cosec \Rightarrow -cosec\,cot$
$cot \Rightarrow -cosec^2$

We deal with the exponential first since that is the main function, but when we differentiate the power which is part of the exponential differentiation rule, we have to use our trig differentiation rule to do this

$$\frac{dy}{dx} = e^{cosx}(-\sin x)$$

$$\frac{dy}{dx} = (-\sin x)e^{\cos x}$$

$$\boxed{\frac{dy}{dx} = -\sin x\, e^{\cos x}}$$

$$y = (e^{4x} + 5)^6$$

Here we have a mix of 2 types (harder power and exponential)

Recall the rule for harder powers:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

Recall the rule for exponentials:

"copy the entire exponential and then multiply by the derivative of the power"

We deal with the harder power since that is the main function, but when we differentiate inside the bracket which is part of the harder power differentiation rule, we have to use our exponential differentiation rule to do this

$$\frac{dy}{dx} = 6(e^{4x} + 5)^5(4e^{4x})$$

$$\boxed{\frac{dy}{dx} = 24e^{4x}(e^{4x} + 5)^5}$$

$$f(x) = \sqrt{e^{2x} + e^{-2x}}$$

Firstly we need to write this as $f(x) = (e^{2x} + e^{-2x})^{\frac{1}{2}}$

Here we have a mix of 2 types (harder power and exponential)

Recall the rule for harder powers:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

Recall the rule for exponentials:

"copy the entire exponential and then multiply by the derivative of the power"

We deal with the harder power since that is the main function, but when we differentiate inside the bracket which is part of the harder power differentiation rule, we have to use our exponential differentiation rule to do this

$$f'(x) = \frac{1}{2}(e^{2x} + e^{-2x})^{-\frac{1}{2}}(2e^{2x} + (-2)e^{-2x})$$

$$f'(x) = (e^{2x} - e^{-2x})(e^{2x} + e^{-2x})^{-\frac{1}{2}}$$

$$\boxed{f'(x) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}}$$

$$y = \ln(sinx)$$

Here we have a mix of 2 types (log and trig)

Recall the rule for logs:

"This turns into to a fraction which looks like $\frac{derivative\ of\ argument}{copy\ of\ argument}$. Notice how the ln disappears"

Recall the rule for trig:

"Change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle"
Our 6 trig functions that we have to remember are:
$\boldsymbol{sin \Rightarrow cos}$ **(we have this one in this example)**
$cos \Rightarrow -sin$
$tan \Rightarrow sec^2$
$sec \Rightarrow sec\ tan$
$cosec \Rightarrow -cosec\ cot$
$cot \Rightarrow -cosec^2$

We deal with the log first since that is the main function, but when we differentiate inside the argument part which is part of the log differentiation rule, we have to use our trig differentiation rules to do this

$$\frac{dy}{dx} = \frac{\cos x}{\sin x}$$

$$\boxed{\frac{dy}{dx} = \cot x}$$

$$y = \ln(1-2x)^3$$

Here we have a mix of 2 types (log and harder powers)

Recall the rule for logs:

"This turns into to a fraction which looks like $\frac{derivative\ of\ argument}{copy\ of\ argument}$. Notice how the ln disappears"

Recall the rule for harder powers:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

We deal with the log first since that is the main function, but when we differentiate inside the argument part which is part of the log differentiation rule, we have to use our harder power differentiation rules to do this

$$\frac{dy}{dx} = \frac{3(1-2x)^2(-2)}{(1-2x)^3}$$

$$\frac{dy}{dx} = \frac{-6(1-2x)^2}{(1-2x)^3}$$

$$\boxed{\frac{dy}{dx} = -\frac{6}{1-2x}}$$

$$f(x) = \sin^3 4x$$

This is one that students so often get wrong.

We have to first write the trig to a power in a more familiar way

$$f(x) = \sin^3 4x = (\sin 4x)^3$$

Here we have a mix of 2 types (harder power and trig)

Recall the rule for harder powers:

"bring the power down to the front (and multiply it by the number at the front if there is one), then subtract one from the power (keeping what is inside the bracket the same) and then multiply by the derivative of what is inside the bracket"

Recall the rule for trig:

"Change the trig function to what it is meant to go to (keep the angle the same) and then multiply by the derivative of the angle"
Our 6 trig functions that we have to remember are:
$sin \Rightarrow cos$ (we have this one in this example)
$cos \Rightarrow -sin$
$tan \Rightarrow sec^2$
$sec \Rightarrow sec\ tan$
$cosec \Rightarrow -cosec\ cot$
$cot \Rightarrow -cosec^2$

We deal with the harder power since that is the main function, but when we differentiate inside the bracket which is part of the harder power differentiation rule, we have to use our trig differentiation rule to do this

$$f'(x) = 3(\sin 4x)^2\,(4\cos 4x)$$

$$f'(x) = 12\cos 4x\,(\sin 4x)^2$$

$$f'(x) = 12\cos 4x \sin^2 4x$$

Product and Quotient Rule Examples

$$y = 2x(x^2-1)^5. \text{ Find } \frac{dy}{dx}.$$

$$y = 2x(x^2-1)^5$$

We must use **product rule** here since we have one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) **multiplied together** (here we have an easy power and a harder power).

Important: People often fall into the trap of not thinking that product rule is necessary here, because they fail to realise that the easy power counts as a type so we have multiplication of 2 of the types.

Way 1: Use the formula $y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$y = 2x(x^2-1)^5$$

Note: We call one function u and the other v. It doesn't matter which we call u or v since the formula is

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

We are just multiplying and then adding the two multiplications. Multiplication can be done in any order and so can the addition be done in any order afterwards.

Let's call the pink function u and the blue function v

$$u = 2x, v = (x^2 - 1)^5$$

We differentiate each

$$\frac{du}{dx} = 2, \frac{dv}{dx} = 5(x^2 - 1)^4\,(2x)$$

Plug into the formula $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = 2x(5)(x^2 - 1)^4(2x) + (x^2 - 1)^5(2)$$

Simplify by re-ordering the constants from each term. Let's colour code to explain this.

$$\frac{dy}{dx} = 2x(5)(x^2 - 1)^4(2x) + (x^2 - 1)^5(2)$$

$$\frac{dy}{dx} = 20x^2(x^2 - 1)^4 + 2(x^2 - 1)^5$$

Way 2: Understand what formula is telling us

$$y = 2x(x^2 - 1)^5$$

The formula basically says in English, differentiate one function at a time

(**differentiate** 1st function)(copy 2nd function)+(copy 1st function)(**differentiate** 2nd function)

$$\frac{dy}{dx} = 2(x^2 - 1)^5 + 2x(5)(x^2 - 1)^4(2x)$$

simplify

$$\boxed{\frac{dy}{dx} = 20x^2(x^2 - 1)^4 + 2(x^2 - 1)^5}$$

$y = x^2e^x$. Find $\frac{dy}{dx}$.

$$y = x^2e^x$$

We must use **product rule** here since we have one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) **multiplied together** (here we have an easy power and an exponential).

Way 1: Use the formula $y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$y = x^2e^x$$

Note: We call one function u and the other v. It doesn't matter which we call u or v since the formula is

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

We are just multiplying and then adding the two multiplications. Multiplication can be done in any order and so can the addition be done in any order afterwards.

Let's call the pink function u and the blue function v

$$u = x^2, v = e^x$$

We differentiate each

$$\frac{du}{dx} = 2x, \frac{dv}{dx} = e^x$$

Plug into the formula $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = (x^2)(e^x) + e^x(2x)$$

Simplify by re-ordering the $2x$ in the second term

$$\frac{dy}{dx} = x^2e^x + 2xe^x$$

Way 2: Understand what formula is telling us

$$y = x^2e^x$$

The formula basically says in English, differentiate one function at a time

(**differentiate** 1st function)(copy 2nd function)+(copy 1st function)(**differentiate** 2nd function)

$$\frac{dy}{dx} = (2x)e^x + (x^2)(e^x)$$

Simplify

$$\boxed{\frac{dy}{dx} = 2xe^x + x^2e^x}$$

$$y = \frac{3x+1}{2x+1}. \text{ Find } \frac{dy}{dx}.$$

$$y = \frac{3x+1}{2x+1}$$

We must use quotient rule here since we have **division** of one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) (here we have an easy power and an easy power type)

Way 1: Use the formula $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$y = \frac{3x+1}{2x+1}$$

Note: We call one function u and the other v.

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$$

This time it MATTERS which we call u or v since we are subtracting and the denominator of the original fraction is what must be squared, not the numerator. Subtraction cannot be done in any order. For example, $4 - 2$ is not the same as $2 - 4$

So we must call u the numerator (pink function) and v the denominator (blue function)

$$u = 3x + 1, v = 2x + 1$$

We differentiate each

$$\frac{du}{dx} = 3, \frac{dv}{dx} = 2$$

Plug into the formula $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx}=\frac{(2x+1)(3)-(3x+1)(2)}{(2x+1)^2}$$

Simplify the numerator

$$\frac{dy}{dx}=\frac{(6x+3)-(6x+2)}{(2x+1)^2}$$

$$\frac{dy}{dx}=\frac{1}{(2x+1)^2}$$

Way 2: Understand what formula is telling us

$$y=\frac{3x+1}{2x+1}$$

The formula basically says in English:

$$\frac{(copy\ denominator)(\boldsymbol{differentiate}\ numerator)-(copy\ numerator)(\boldsymbol{differentiate}\ denominator)}{(denominator)^2}$$

$$\frac{dy}{dx}=\frac{(2x+1)(3)-(3x+1)(2)}{(2x+1)^2}$$

Simplify the numerator

$$\frac{dy}{dx}=\frac{(6x+3)-(6x+2)}{(2x+1)^2}$$

$$\boxed{\frac{dy}{dx}=\frac{1}{(2x+1)^2}}$$

$y=\frac{x^2}{\ln x}$. Find $\frac{dy}{dx}$

$$y=\frac{x^2}{\ln x}$$

We must use quotient rule here since we have **division** of one of the 6 differentiation types (easy power, harder power, ln, exponential, trig, inverse trig) (here we have an easy power and a log type)

Way 1: Use the formula $y=\frac{u}{v}\Rightarrow\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$$y=\frac{x^2}{\ln x}$$

Note: We call one function u and the other v.

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

This time it MATTERS which we call u or v since we are subtracting and the denominator of the original fraction is what must be squared, not the numerator. Subtraction cannot be done in any order. For example, $4-2$ is not the same as $2-4$

So we must call u the numerator (pink function) and v the denominator (blue function)

$$u=x^2, v=\ln x$$

We differentiate each

$$\frac{du}{dx}=2x,\frac{dv}{dx}=\frac{1}{x}$$

Plug into the formula $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\ln x\,(2x) - x^2\left(\frac{1}{x}\right)}{(\ln x)^2}$$

Simplify the numerator

$$\frac{dy}{dx} = \frac{2x\ln x - x}{(\ln x)^2}$$

Way 2: Understand what formula is telling us

$$y = \frac{x^2}{\ln x}$$

The formula basically says in English:

$$\frac{(copy\ denominator)(\boldsymbol{differentiate}\ numerator) - (copy\ numerator)(\boldsymbol{differentiate}\ denominator)}{(denominator)^2}$$

$$\frac{dy}{dx} = \frac{(\ln x)(2x) - x^2\left(\frac{1}{x}\right)}{(\ln x)^2}$$

Simplify the numerator

$$\boxed{\frac{dy}{dx} = \frac{2x\ln x - x}{(\ln x)^2}}$$

Product and Quotient - Getting Into Certain Forms

$y = (x+1)^4(2x-2)^5$. Show that $\frac{dy}{dx} = 2(9x+1)(x+1)^3(2x-2)^4$

$$y = (x+1)^4(2x-2)^5$$

We must use product rule

$$\frac{dy}{dx} = (x+1)^4(5)(2x-2)^4(2) + (2x-2)^5(4)(x+1)^3$$

Simplify by multiplying constants and re-ordering. Let's colour code this for ease of explanation.

$$\frac{dy}{dx} = (x+1)^4(5)(2x-2)^4(2) + (2x-2)^5(4)(x+1)^3$$

$$\frac{dy}{dx} = 10(x+1)^4(2x-2)^4 + 4(2x-2)^5(x+1)^3$$

To simplify further and get into the required form we must factorise by taking out what is common to both terms. Let's colour code this again for ease of explanation.

$$\frac{dy}{dx} = 10(x+1)^4(2x-2)^4 + 4(x+1)^3(2x-2)^5$$

Take out the HCF of the numbers
Take out the HCF of the pink terms (lowest power of each)
Take out the HCF of the blue terms (lowest power of each)

$$= 2(x+1)^3(2x-2)^4[5(x+1) + 2(2x-2)]$$

Notice how we subtracted the powers in order to get the powers of the terms inside the square bracket (or asked ourselves what power we need to add to the power we have outside the bracket to end up with the power we want)

Simplify what is inside the square bracket

$$= 2(x+1)^3(2x-2)^4[5x + 5 + 4x - 4]$$

Simplify again

$$= 2(x+1)^3(2x-2)^4(9x+1)$$

$$= 2(9x+1)(x+1)^3(2x-2)^4$$

$y = \frac{x^2-4x+12}{(x-3)^2}$. Show that $\frac{dy}{dx} = -\frac{2(x+6)}{(x-3)^3}$

$$y = \frac{x^2-4x+12}{(x-3)^2}$$

We must use quotient rule

$$\frac{dy}{dx} = \frac{(x-3)^2(2x-4)-(x^2-4x+12)(2)(x-3)}{(x-3)^4}$$

Re-order: Bring the constant of 2 to the front of the second term in the numerator

$$\frac{dy}{dx} = \frac{(x-3)^2(2x-4)-2(x-3)(x^2-4x+12)}{(x-3)^4}$$

We still have not achieved the required form. There are 2 ways to proceed to do this. Since the powers are small and it is easy to simplify we don't have to factorise straight away

Way 1: Factorise the numerator straight away

$$\frac{dy}{dx} = \frac{(x-3)^2(2x-4)-2(x-3)(x^2-4x+12)}{(x-3)^4}$$

Take out the HCF of the pink terms in the numerator (lowest power)

$$\frac{dy}{dx} = \frac{(x-3)[(x-3)(2x-4)-2(x^2-4x+12)]}{(x-3)^4}$$

Simplify what is inside the square bracket

$$\frac{dy}{dx} = \frac{(x-3)[2x^2-10x+12-2x^2+8x-24]}{(x-3)^4}$$

$$\frac{dy}{dx} = \frac{(x-3)[-2x-12]}{(x-3)^4}$$

$$\frac{dy}{dx} = \frac{-2x-12}{(x-3)^3}$$

$$\frac{dy}{dx} = -\frac{2(x+6)}{(x-3)^3}$$

Way 2: Simplify the numerator first

This is not always the best method as it can be hard to factorise the numerator after in order to cancel

$$\frac{dy}{dx} = \frac{(2x-4)(x-3)-2(x^2-4x+12)}{(x-3)^3}$$

Simplify the numerator by expanding the brackets

$$\frac{dy}{dx} = \frac{2x^2-10x+12-2x^2+8x-24}{(x-3)^3}$$

Collect like terms

$$\frac{dy}{dx} = \frac{-2x-12}{(x-3)^3}$$

Factorise the numerator

$$\frac{dy}{dx} = -\frac{2(x+6)}{(x-3)^3}$$

$y = \frac{\sqrt{x^2+1}}{x-1}$. Show that $\frac{dy}{dx} = \frac{-(x+1)}{(x-1)^2\sqrt{x^2+1}}$

$$y = \frac{\sqrt{x^2+1}}{x-1}$$

Firstly, let's rewrite the numerator using an exponent

$$y = \frac{(x^2+1)^{\frac{1}{2}}}{x-1}$$

We must use the quotient rule

$$\frac{dy}{dx} = \frac{(x-1)\left(\frac{1}{2}\right)(x^2+1)^{-\frac{1}{2}}(2x) - (1)(x^2+1)^{\frac{1}{2}}}{(x-1)^2}$$

Simplify the numerator by multiplying constants in first term and rewriting $(x^2+1)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{\frac{x(x-1)}{\sqrt{x^2+1}} - \sqrt{x^2+1}}{(x-1)^2}$$

Way 1:

Work on each of the numerators and denominators separately

$$\frac{dy}{dx} = \frac{\frac{x(x-1)}{\sqrt{x^2+1}} - \sqrt{x^2+1}}{(x-1)^2}$$

We get a common denominator in the numerator

$$\frac{dy}{dx} = \frac{\frac{x(x-1)}{\sqrt{x^2+1}} - \frac{x^2+1}{\sqrt{x^2+1}}}{(x-1)^2}$$

Combine fractions in the numerator

$$\frac{dy}{dx} = \frac{\frac{x(x-1)-(x^2+1)}{\sqrt{x^2+1}}}{(x-1)^2}$$

Simplify the numerator

$$\frac{dy}{dx} = \frac{\frac{-x-1}{\sqrt{x^2+1}}}{(x-1)^2}$$

Rewrite the fraction as numerator ÷ denominator

$$\frac{dy}{dx} = \frac{-x-1}{\sqrt{x^2+1}} \div (x-1)^2$$

"keep change flip"

$$\frac{dy}{dx} = \frac{-x-1}{\sqrt{x^2+1}} \times \frac{1}{x+\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{-(x+1)}{(x-1)^2\sqrt{x^2+1}}$$

Way 2:

Multiply every term in the numerator and denominator by $\sqrt{x^2+1}$ to kill the fractions

Note: This is just multiplying through by $\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = \frac{\frac{x(x-1)}{\sqrt{x^2+1}} \times \sqrt{x^2+1} - \sqrt{x^2+1} \times \sqrt{x^2+1}}{(x-1)^2\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{x(x-1)-(x^2+1)}{(x-1)^2\sqrt{x^2+1}}$$

Expand the numerator

$$\frac{dy}{dx} = \frac{x^2-x-x^2-1}{(x-1)^2\sqrt{x^2+1}}$$

Combine like terms in the numerator

$$\frac{dy}{dx} = \frac{-(x+1)}{(x-1)^2\sqrt{x^2+1}}$$

$y = \ln(x + \sqrt{x^2+1})$. Show that $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

We must use the log differentiation rule. This turns into to a fraction which looks like $\frac{derivative\ of\ argument}{copy\ of\ argument}$.

$$\frac{dy}{dx} = \frac{1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)}{x + \sqrt{x^2+1}}$$

Multiplying the constants in the second term in the numerator and rewrite $\sqrt{x^2+1}$ as $(x^2+1)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

Way 1:

Work on each of the numerators and denominators separately

$$\frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

We get a common denominator in the numerator

$$\frac{dy}{dx} = \frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

Combine fractions in the numerator

$$\frac{dy}{dx} = \frac{\frac{x + \sqrt{x^2+1}}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

Rewrite the fraction as numerator ÷ denominator

$$\frac{dy}{dx} = \frac{x + \sqrt{x^2+1}}{\sqrt{x^2+1}} \div x + \sqrt{x^2+1}$$

"keep change flip"

$$\frac{dy}{dx} = \frac{\cancel{x + \sqrt{x^2+1}}}{\sqrt{x^2+1}} \times \frac{1}{\cancel{x + \sqrt{x^2+1}}}$$

"Cancel" common factors

$$\frac{dy}{dx} = \frac{\cancel{x + \sqrt{x^2+1}}}{\sqrt{x^2+1}} \times \frac{1}{\cancel{x + \sqrt{x^2+1}}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

Way 2:

Multiply ALL terms by $\sqrt{x^2+1}$ to "kill" the fraction

$$\frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{1\sqrt{x^2+1} + \frac{x}{\sqrt{x^2+1}}\sqrt{x^2+1}}{x\sqrt{x^2+1} + \sqrt{x^2+1}\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}(x + \sqrt{x^2+1})}$$

"Cancel" common factors

$$\frac{dy}{dx} = \frac{\cancel{\sqrt{x^2+1} + x}}{\sqrt{x^2+1}\cancel{(x + \sqrt{x^2+1})}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

$y = \frac{x-1}{\sqrt{x+1}}$ Show that $\frac{dy}{dx} = \frac{x+c}{k\sqrt{(x+1)^p}}$, where $c, k, p \in \mathbb{N}$

$$y = \frac{x-1}{(x+1)^{\frac{1}{2}}}$$

Use Quotient Rule

$$\frac{dy}{dx} = \frac{(x+1)^{\frac{1}{2}}(1) - (x-1)\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}}}{x+1}$$

Reorder the constants in the numerator

$$\frac{dy}{dx} = \frac{(x+1)^{\frac{1}{2}} - \frac{1}{2}(x-1)(x+1)^{-\frac{1}{2}}}{x+1}$$

To get the form that the answer wants we need to factorise the numerator by taking out what is common, which is $x+1$ and the lowest power is $-\frac{1}{2}$

$$\frac{dy}{dx} = \frac{(x+1)^{-\frac{1}{2}}\left[(x+1) - \frac{1}{2}(x-1)\right]}{x+1}$$

Notice how we subtracted the powers in order to get the powers of the terms inside the square bracket (or asked ourselves what power we need to add to the power we have outside the bracket to end up with the power we want)

Simplify what is inside the square brackets

$$\frac{dy}{dx} = \frac{(x+1)^{-\frac{1}{2}}\left[x+1-\frac{1}{2}x+\frac{1}{2}\right]}{x+1}$$

Simplify again

$$\frac{dy}{dx} = \frac{(x+1)^{-\frac{1}{2}}\left[\frac{1}{2}x+\frac{3}{2}\right]}{x+1}$$

Use indices rules on the common terms. The best way to think of this is that anytime we move a term between the numerator and denominator we change the sign of the power

$$\frac{dy}{dx} = \frac{\frac{1}{2}x+\frac{3}{2}}{(x+1)\,(x+1)^{-\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x+\frac{3}{2}}{(x+1)^{1+\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x+\frac{3}{2}}{(x+1)^{\frac{3}{2}}}$$

Use indices rule $x^{\frac{a}{b}} = \sqrt[b]{x^a}$. Note when $b = 2$ we don't write it.

$$\frac{dy}{dx} = \frac{\frac{1}{2}x+\frac{3}{2}}{\sqrt{(x+1)^3}}$$

We multiply all terms by 2 to get rid of the fractions i.e. we multiply by $\frac{2}{2}$

$$\frac{dy}{dx} = \frac{x+3}{2\sqrt{(x+1)^3}}$$

$$\boxed{\therefore c = 3, k = 2, p = 3}$$