AP Calculus Formulae Sheet

Shapes

Area of Triangle	$\frac{1}{2}$ x base x height
Area of Parallelogram	base x height
Area of Trapezoid	$\frac{1}{2}$(sum of parallel sides) x height
Circumference & Area: Circle	$c = 2\pi r,\ A = \pi r^2$
Cuboid Surface area	$SA = 2xy + 2xz + 2yz$ where $x, y, and\ z$ are side lengths
Cuboid Volume	$V = xyz$ where $x, y, and\ z$ are side lengths
Cylinder Surface Area	$SA = 2\pi rh + 2\pi r^2$ Note: Curved part: $2\pi rh$
Cylinder Volume	$V = \pi r^2 h$
Cone Surface Area	$SA = \pi rl + \pi r^2$ Note: Curved part: πrl, where l is slant length
Cone Volume	$V = \frac{1}{3}\pi r^2 h$
Sphere Surface Area	$SA = 4\pi r^2$ (Hemisphere= $3\pi r^2$)
Sphere Volume	$v = \frac{4}{3}\pi r^3$ (Hemisphere= $\frac{2}{3}\pi r^3$)
Prism Volume	V =Area of cross section x height
Pyramid Volume	$V = \frac{1}{3} \times base\ area \times h$

Indices

Multiplication	$x^a \times x^b = x^{a+b}$ $(cx^a y^b)^d = c^d x^{ad} y^{bd}$
Division	$x^a \div x^b = \frac{x^a}{x^b} = x^{a-b}$
Negative Powers	$x^{-n} = \frac{1}{x^n}$
Fractions	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ and $\left(\frac{x}{y}\right)^{-n} = \frac{y^n}{x^n}$
Rational Powers	$a^{\frac{n}{m}} = (\sqrt[m]{a})^n$

Binomial

Binomial Theorem: integer powers	$(a+b)^n$ $=a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$
Binomial Coefficient	$\binom{n}{r} = nc_r = \frac{n!}{(n-r)!\,r!}$

Geometry

Straight Line: Equation (gradient means slope) ∥ same slope, ⊥ "flip fraction, change sign" which means slopes × to make −1)	• Slope intercept form: $y = mx + c$ • General form: $ax + by + d = 0$ • Point slope form: $y - y_1 = m(x - x_1)$
Straight Line: Gradient	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Distance between $(x_1, y_1), (x_2, y_2)$	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Coordinates of midpoint of $(x_1, y_1), (x_2, y_2)$	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Quadratics

Quadratic Function: Solutions to $ax^2 + bx + c = 0$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$
Quadratic Function: Axis of Symmetry	$f(x) = x^2 + bx + c \Rightarrow x = -\frac{b}{2a}$
Quadratic Function: Discriminant	$\Delta = b^2 - 4ac$ • > 0 (2 real distinct roots) • $= 0$ (2real repeated/double roots) • < 0 (no real roots)
Completing The Square $ax^2 \pm bx + c = 0$	$a\left(x \pm \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$
Exponentials & Logarithm Rules`	• $c\log_a b \Leftrightarrow \log_a b^c$ • $\log_a b = c \Leftrightarrow a^c = b$, $a, b, > 0, a \neq 1$ • $\log_a b + \log_a c \Leftrightarrow \log_a bc$ • $\log_a b - \log_a c \Leftrightarrow \log_a \frac{b}{c}$ • $\log_a b \Leftrightarrow \frac{\log_c b}{\log_c a}$ • Solving a power of x: log both sides if 2 terms or use substitution if 3 terms • Solving an exponential : In both sides • Solving a logarithm: raise e both sides or write as $\log_e$ as procced as for log

Dealing with Inequalities

Polynomials, Rational	Use number line to put zeros and undefined points and check signs either side
Mod	$\lvert x\rvert < a \Rightarrow -a < x < a$ $\lvert x\rvert > a \Rightarrow x < -a\ OR\ x > a$ OR: graph each and then see where one graph lies above (>)/below (<) the other

Limits

Graphically: Can we trace inwards from the left and right and still reach the same y coordinate? If yes, has a limit

Method:
Direct substitution, if get a number or one of the following 4 then done:

1) $\frac{Any\ non\ zero\ number}{0} = undefined$ 2) $\frac{\pm\infty}{non\ infinte\ number} = \pm\infty$

3) $\frac{0}{Any\ non\ zero\ number} = 0$ 4) $\frac{Any\ non\ infinte\ number}{\pm\infty} = 0$

1) and 2) just say that there is no limit and 3) and 4) just say limit is zero

If not, and get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ it is indeterminate form and not an answer. We then either

- Factorise and cancel
- Rationalise and cancel
- Use a trig identity and cancel
- Apply L' Hopital's Rule (differentiate numerator and denominator)

and then apply substitution again

Other hints:
If trig and → 0 : use small angle approx. or identity and simplify or squeeze theorem
If algebraic and → ∞: divide by highest power in denominator or memorise that

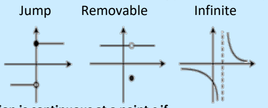
- Even powers in numerator and denominator
- Bottom heavy $y = 0$
- Top Heavy $y = \infty$ (no asymptote)

Note: Watch out for when you have roots in the denominator.
If all the above has failed use squeeze theorem!

Continuity

Graphically: Can we trace the curve without taking pen off page? If yes, continuous!

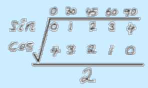
Jump Removable Infinite



Definition: A function is continuous at a point c if

1) $f(c) is\ defined$, meaning the function has a value at $x = c$ i.e. when you plug c in the function it returns a value
2) $\lim_{x\to c} f(x)\ exists$ $i.e, \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$, meaning the limit exists at x=c (i.e. the two sided limits are equal)
3) $\lim_{x\to c} f(x) = f(c)$, meaning the value of the function at x=c is equal to the value of the limit at $x = c$

Trigonometry

Sine Rule	Finding a side: $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$ Finding an angle: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
Cosine Rule	inding a side: $a^2 = b^2 + c^2 - 2bc\cos A$ Finding an angle:$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$
Area of Triangle	$\frac{1}{2}absinC$
Degrees to radians and vice versa	D to R: $\times\frac{\pi}{180}$ R to D: $\times\frac{180}{\pi}$
Length of an arc	$\frac{\theta}{360} \times 2\pi r$ (degrees) or $r\theta$ (radians)
Area of a Sector	$\frac{\theta}{360} \times \pi r^2$ (degrees) or $\frac{1}{2}r^2\theta$ (radians)
Small Angle Approximations	$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$
Pythagorean identity 1	$sin^2 x + cos^2 x = 1$
Pythagorean identity 2	$1 + tan^2 x = sec^2 x.$
Pythagorean identity 3	$1 + cot^2 x = cosec^2 x$
Cofunction	$\cos x = \sin(90 - x)$ $\sin x = \cos(90 - x)$
Identity of tan x	$\tan x = \frac{\sin x}{\cos x}$
Reciprocal	$\sec x = \frac{1}{\cos x}, \csc x = \frac{1}{\sin x}, \cot x = \frac{1}{\tan x}$
Double Angle	$\sin 2x = 2\sin x\cos x$ $cos2x = cos^2 x - sin^2 x$ $= 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{\cos 2x + 1}{2}$ $= 1 - 2\sin^2\theta \Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$ $\tan 2x = \frac{2\tan x}{1 - tan^2 x}$
Half Angle	$sin\ \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$ $cos\ \frac{x}{2} = \pm\sqrt{\frac{\cos x + 1}{2}}$
Compound Angle	$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ $cos(A \pm B) = cosAcosB \mp sinAsinB$ $tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
Special Angles	
Even/Odd	$\sin(-x) = -\sin x$, $cos(-x) = \cos x$, $\tan(-x) = -\tan x$

Averages

Average value of function f on [a,b]	$\frac{1}{b-a}\int_a^b f(x)dx$
Average rate of function of f on [a,b]	$\frac{f(b) - f(a)}{b-a}$
Instantaneous rate at $x = c$ (Note: at a point, not an interval)	$f'(c)$

Functions

Inverse	Replace $f(x)$ with y, swap x & y, solve for y
Composite	$fg(x)$ means plug $g(x)$ into $f(x)$
Transformations: $af(bx + c) + d$ "anything in a bracket does the opposite"	a=vertical stretch sf a, b=horizontal stretch sf $\frac{1}{b}$ c=translation c units x direction, d=translation d units in y direction $f(-x)$=reflc in $y\ axis$, $-f(x)$=reflc in $y\ axis$
Inverse	Replace $f(x)$ with y, swap x & y, solve for y
Odd/Even	Even: $f(-x) = f(x)$, odd: $f(-x) = -f(x)$
Periodic	$f(x + p) = f(x)$ where p is the period
Basic Domain "the x values allowed" (see table below for domain and range for all common functions in more detail)	Fractions $\frac{\cdots}{\cdots}$: $x \in \mathbb{R}$, $x \neq$ value(s) where denom $= 0$ Roots: $\sqrt{\ldots}$: Solve for part under root to be ≥ 0 Exponentials $e^{\cdots}$: $x \in \mathbb{R}$ (power can be anything, no restriction on it) Logarithms $\ln(\ \ldots)$: Solve for argument to be > 0

Linear: $y = mx + c$
Domain: x∈ $\mathbb{R}$
Range: y∈ $\mathbb{R}$

Quadratic: $y = \pm a(bx + c)^2$+ d
Domain: x∈ $\mathbb{R}$
Range: $y \geq d$ if min, $y \leq d$ if max

Exponential: $y = ae^{bx+c} + d$
Domain: x∈ $\mathbb{R}$ (Hint: power of exp can be anything, so no restriction)
Range: $y > d$ if $a > 0$, $y < d$ if $a < 0$ (Hint: exp can't be zero)

Logarithm: $y = aln(bx + c)$+d
Domain: $x > -\frac{c}{b}$ (Hint: ln can't take a neg number so $bx + c > 0$)
Range: y ∈ $\mathbb{R}$
Asymptote: $x = -\frac{c}{b}$

Root: $a\sqrt{bx + c} + d$:
Domain: $x \geq -\frac{c}{b}$ (Hint: under root must be positive so b$x + c$ ≥0)
Range: $y \geq d$ if $a > 0$ and $y \leq d$ if a<0

Modulus $a\lvert bx + c\rvert + d$:
Domain: x∈ $\mathbb{R}$
Range: $y \geq d$ if $a > 0$ and $y \leq d$ if a <0

Note: Definition of $\lvert x\rvert = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

Rational: $\frac{ax+b}{cx+d} + e$
Domain: x∈ $\mathbb{R}$, $x \neq -\frac{d}{c}$ (Hint:denom≠0)
Range: y∈ $\mathbb{R}$, $y \neq \frac{a}{c} + e$
Asymptotes: $x = -\frac{d}{c}, y = \frac{a}{c} + e$
Note: often a and or e are zero

Trigonometry:$y = asin(bx + c)$+d
$y = acos(bx + c)$+d
Domain: $x \in \mathbb{R}$
Range: $-a + d \leq y \leq a + d$
Note: If asked to find values of a,b,c,d
a = amplitude=$\frac{\max y - \min y}{2}$
$b = \frac{2\pi}{period}$ or $\frac{360}{period}$
d = principal axis=$\frac{\max y + \min y}{2}$
c =phase shift (plug in point to find after finding a, b and d)

Trigonometry:$y = atan(bx + c)$+d
Domain: $x \in \mathbb{R}, x \neq \frac{\pi}{2} + n\pi$
Range:$-\infty \leq y \leq \infty$

Inverse trig: $y = sin^{-1}x$
Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Inverse trig: $y = cos^{-1}x$
Domain: $-1 \leq x \leq 1$
Range: $0 \leq x \leq \pi$

Inverse trig: $y = tan^{-1}x$
Domain: $-\infty \leq x \leq \infty$
Range: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Graphing harder rational functions

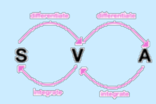
- Vertical asymptotes set denominator =0 and solve
- Horizontal asymptotes: Find $\lim_{x\to\infty} f(x)$
  Easiest method: Can just use the fact that even powers y =coefficents of ratio of highest powers, Bottom heavy $y = 0$, Top Heavy $y = \infty$ (no asymptote)
- Slant: only exists if top heavy. Divide and quotient part is slant asymptote
- Intercepts : set $x = 0$, find y and vice versa
- check the behaviour each side of vertical asymptote $x = a$: $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.
  We want to know whether y tends to tend to $+\infty$ or $-\infty$.
  Easiest Method: Can just try a value just less than and just bigger than a
- check the behaviour near horizontal asymptote: $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ to find which side of the horizontal asymptotes you're on when far out to left or right.
  Easiest Method: You don't have to find the limit here, you can just use the other features of the graph that you already have to help you

Remember: You can NEVER cross a vertical asymptote, but can cross a horizontal one centrally.
We cannot cross a horizontal one far out though

Differentiation

Turning/Stationary Points (Max/Min/Extrema)	Solve $\frac{dy}{dx} = 0$ Remember to include values where derivative is undefined too i.e. the holes Absolute: check endpoint values too. Plug all x values into function and see which gives greatest value
Proving whether Max/Min	If $\frac{d^2y}{dx^2} > 0$ min and $\frac{d^2y}{dx^2} < 0$ max Or can do sign change test for $\frac{dy}{dx}$
When doesn't derivative not exist?	• Corner (sharp turn) $y = \lvert x\rvert$ • Vertical Tangent $y = x^{\frac{1}{3}}$ • Cusps "concave corners" $y = x^{\frac{2}{3}}$ • Discontinuities $f(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$
Points of Inflection	solve $\frac{d^2y}{dx^2} = 0$
Increasing/Decreasing	Increasing: solve $\frac{dy}{dx} > 0$ decreasing: solve $\frac{dy}{dx} < 0$
Convex/Concave	concave up/convex: solve $\frac{d^2y}{dx^2} > 0$ concave down/concave: solve $\frac{d^2y}{dx^2} < 0$
Differentiation 1st Principles	• Generally: $\frac{dy}{dx} = f'(x) = \lim_{h\to0}\frac{f(x+h) - f(x)}{h}$ • At a point a: $f'(a) = \lim_{h\to0}\frac{f(a+h) - f(a)}{h}$ • Alternate form: $f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a}$ (we don't really use this) • To show derivative exists: show 2 sides limits equal $\lim_{h\to0^-}\frac{f(a+h) - f(a)}{h} = \lim_{h\to0^+}\frac{f(a+h) - f(a)}{h}$ Differentiability ⟹Continuity Continuity ⇏Differentiability
Chain Rule	$y = g(u), u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
Product Rule	$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Implicit	"every time we differentiate a y we write $\frac{dy}{dx}$"
Derivatives	• $x^n \Rightarrow nx^{n-1}$ • $(f(x))^n \Rightarrow n(f(x))^{n-1}f'(x)$ • $\ln(f(x)) \Rightarrow \frac{f'(x)}{f(x)}$ • $\sin f(x) \Rightarrow f'(x)\cos f(x)$ • $\cos f(x) \Rightarrow -f'(x)\sin f(x)$ • $e^{f(x)} \Rightarrow f'(x)e^{f(x)}$ • $a^{f(x)} \Rightarrow f'(x)\,a^{f(x)}\ln a$ • $\tan f(x) \Rightarrow f'(x)\sec^2 f(x)$ • $\sec f(x) \Rightarrow f'(x)\sec f(x)\tan f(x)$ • $cosec\ f(x) \Rightarrow -f'(x)cosec\ f(x)\cot f(x)$ • $\cot f(x) \Rightarrow -f'(x)csc^2 f(x)$ • $\sin^{-1} f(x) \Rightarrow \frac{f'(x)}{\sqrt{1-(f(x))^2}}$ • $\cos^{-1} f(x) \Rightarrow -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$ • $\tan^{-1} f(x) \Rightarrow \frac{f'(x)}{1+(f(x))^2}$ • $\sec^{-1} f(x) \Rightarrow \frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$ • $cosec^{-1} f(x) \Rightarrow -\frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$ • $\cot^{-1} f(x) \Rightarrow -\frac{f'(x)}{1+(f(x))^2}$
IVT	If f is continuous over [a,b] and w is a number between f(a) and f(b) then ∃c ∈ [a,b] s.t. $f(c) = w$ This is mainly used to show that a certain value of a function exists
MVT	If f is continuous on [a,b] and differentiable on (a,b) then ∃ at least one c ∈ [a,b] such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ This is mainly used to show that a certain value of a derivative exists or to find the value when given a table and not the equation
Rolle's Theorem (particular case of MVT)	If f is continuous on [a,b] and differentiable on (a,b) then ∃ at least one c ∈ [a,b] such that $f'(c)$ =0.
Tangents and Normals	$y - y_1 = m(x - x_1)$ Differentiate to get m (tangent ∥, Normal ⊥)
Local linear approx	This is basically just asking for the tangent line

Integration

Area between	curve & $x\ axis$: $\int_{x=a}^{x=b} y\,dx$ curve & y axis: $\int_{y=a}^{y=b} x\,dy$ (take + answer if neg) Between 2 curves: $\int_{x=a}^{x=b}$(top curve-bottom curve)dx Remember to split up if separate areas
Properties	$\int_a^a f(x)dx = 0$ Can swap limits: $\int_a^b f(x)dx = -\int_b^a f(x)dx$ Can split up: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
Integrals	• $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$ • $\int \frac{1}{kx}dx = \frac{1}{k}\ln\lvert x\rvert + c$ • $\int \sin kx\,dx = -\frac{1}{k}\cos kx + c$ • $\int \cos kx\,dx = \frac{1}{k}\sin kx + c$ • $\int e^{kx}dx = \frac{1}{k}e^{kx} + c$ • $\int a^{kx}dx = \frac{1}{k\ln a}a^{kx} + c$ • $\int \sec^2 kx\,dx = \frac{1}{k}\tan kx + c$ • $\int \sec kx\tan kx\,dx = \frac{1}{k}\sec kx + c$ • $\int cosec\,kx\cot kx\,dx = -\frac{1}{k}cosec\,kx + c$ • $\int cosec^2\,kx\,dx = -\frac{1}{k}\cot kx + c$ • $\int \sec kx\,dx = \frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert + c$ • $\int cosec\,kx\,dx = -\frac{1}{k}\ln\lvert cosec kx + \cot kx\rvert$+c • $\int \frac{1}{\sqrt{a^2-(bx)^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + c$ • $\int -\frac{1}{\sqrt{a^2-(bx)^2}}dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + c$ • $\int \frac{1}{a^2+(bx)^2}dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$
Trapezium Rule	$\frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + \cdots) + y_n]$ h = $\frac{b-a}{number\ of\ strips}$
Riemann	Left: $h[y_0 + y_1 + y_2 + y_3 + \cdots y_{n-1}]$ h = $\frac{b-a}{number\ of\ strips}$ Right: $h[y_1 + y_2 + y_3 + y_4 + \cdots y_n]$
Midpoint h = $\frac{b-a}{number\ of\ strips}$	Midpoint:$h\left[y_{\frac{1}{2}} + y_{\frac{3}{2}} + y_{\frac{5}{2}} + \cdots + y_{n-\frac{3}{2}} + y_{n-\frac{1}{2}}\right]$
Kinematics: 	Distance=$\int_{t_1}^{t_2}\lvert v(t)\rvert dt$, Displacement=$\int_{t_1}^{t_2} v(t)dt$ Velocity: $\int_{t_1}^{t_2} a(t)dt$ or $\frac{ds}{dt}$, Acceleration=$\frac{dv}{dt} = \frac{d^2s}{dt^2}$ Changed direction: $v = 0$, Moving right/up ⟹ $v > 0$ Moving left/down : $v < 0$ Speeding up/velocity increasing/accel: a, v same sign Slowing down/velocity decreasing/dec: a, v opp sign Max displacement/max distance from/furthest left or right or up or down: $v = 0$ Min/max velocity : $v' = a = 0$ and then plug back into v
Arc length	$\int_a^b \sqrt{1 + (f'(x))^2}dx$
Volume of revolution	**Disc:** $x\ axis$: $V = \pi\int_{x=a}^{x=b}(radius)^2 dx$ y axis: $V = \pi\int_{y=a}^{y=b}(radius)^2 dy$ **Washer:** About x axis : V = $\pi\int_{x=a}^{x=b}$[(outer radius)^2 − (inner radius)^2)]dx About y axis : V = $\pi\int_{y=a}^{y=b}$[(outer radius)^2 − (inner radius)^2)]dy
Volume of cross sections (⊥to x axis)	$\int_a^b A(x)dx$ where $A(x) = cross\ secitonal\ area\ of\ the\ shape$
Differential equations	Get y's on one side and x's on other and integrate each side. Remember: we only × and÷ to re-arrange
Total Amount	Starting value + $\int_{t_1}^{t_2} rate\ in - \int_{t_1}^{t_2} rate\ out$

Fundamental Theorem

First : $\int_a^b f(x)dx$=F(b) - F(a) i.e.$\int f'(x) = f(x)$

Second: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$