

Quality of tests 8D

1 a $H_0 : \lambda = 6.5$ $H_1 : \lambda < 6.5$

Assume H_0 , so that $X \sim \text{Po}(6.5)$

Significance level 5%, so require c such that $P(X \leq c) < 0.05$

From the Poisson cumulative distribution tables

$$P(X \leq 3) = 0.1118 \text{ and } P(X \leq 2) = 0.0430$$

$P(X \leq 3) > 0.05$ and $P(X \leq 2) < 0.05$ so the critical value is 2

Hence the critical region is $X \leq 2$

$$\text{Size} = P(X \leq 2 \mid \lambda = 6.5) = 0.0430$$

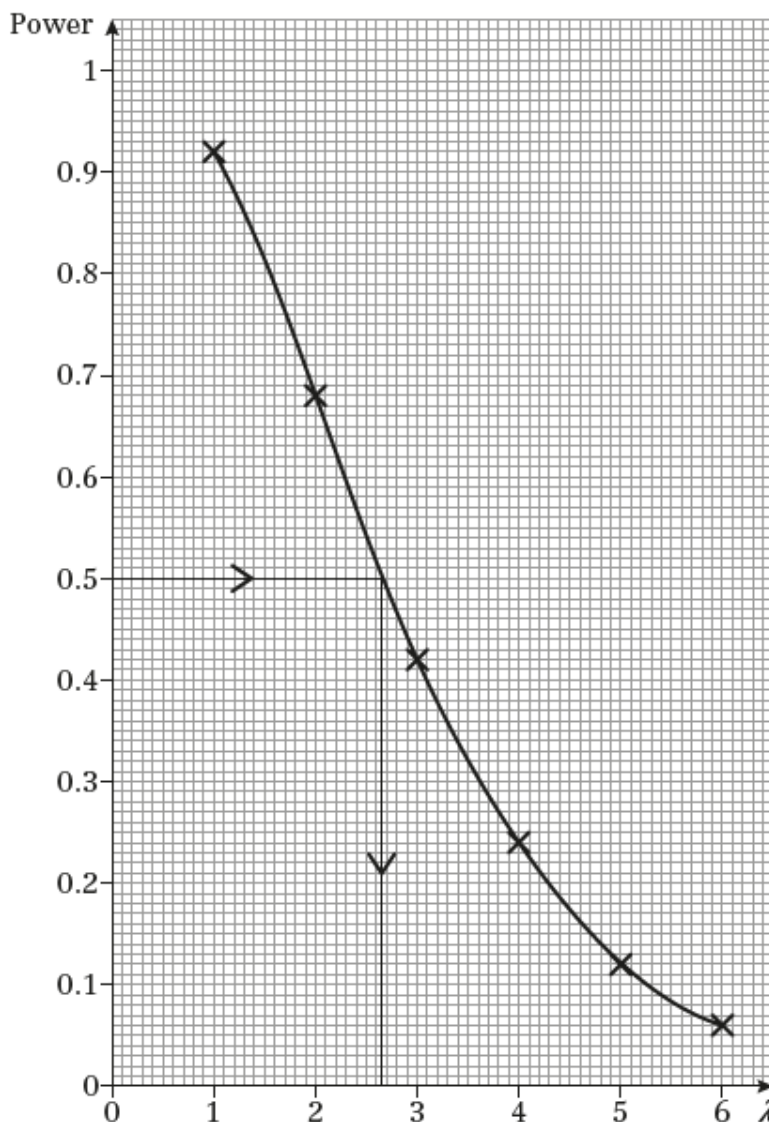
b Power function = $P(X \leq 2 \mid X \sim \text{Po}(\lambda))$

$$= e^{-\lambda} + \frac{e^{-\lambda} \lambda^1}{1!} + \frac{e^{-\lambda} \lambda^2}{2!} = e^{-\lambda} \left(1 + \lambda + \frac{1}{2} \lambda^2 \right)$$

c $\lambda = 2 \Rightarrow s = 5e^{-2} = 0.68$ (2 d.p.)

$$\lambda = 5 \Rightarrow t = \frac{37}{2} e^{-5} = 0.12$$
 (2 d.p.)

d



- 1 e** When $\lambda = 6.5$, the correct conclusion is to accept H_0 . So since size is 0.0430, the probability of accepting $\lambda = 6.5$ is 0.957, which is greater than 0.5. The test is very likely to come to correct conclusion.

When $\lambda < 6.5$, the correct conclusion is to reject H_0 . So require the power of the test > 0.5 , and this can be found by reading from the graph: so the test is more likely than not to come to the correct conclusion for $\lambda < 2.65$

- 2 a** $H_0 : p = 0.45$ $H_1 : p < 0.45$

Critical region $X \leq 2$, where $X \sim B(12, 0.45)$

From the binomial cumulative distribution function tables:

$$\text{Size} = P(X \leq 2) = 0.0421 \text{ (4 d.p.)}$$

- b** Power function = $P(X \leq 2 | X \sim B(12, p))$

$$= P(X = 0 | X \sim B(12, p)) + P(X = 1 | X \sim B(12, p)) + P(X = 2 | X \sim B(12, p))$$

$$= \binom{12}{0} p^0 (1-p)^{12} + \binom{12}{1} p^1 (1-p)^{11} + \binom{12}{2} p^2 (1-p)^{10}$$

$$= (1-p)^{12} + 12p(1-p)^{11} + \frac{12 \times 11}{2} p^2 (1-p)^{10}$$

$$= (1-p)^{12} + 12p(1-p)^{11} + 66p^2(1-p)^{10}$$

- c** Power = $P(X \leq 2 | X \sim B(12, 0.3)) = 0.2528$ (from the tables)

$$\text{Alternatively use the power function, Power} = 0.7^{12} + 3.6 \times 0.7^{11} + 5.94 \times 0.7^{10} = 0.2528 \text{ (4 d.p.)}$$

- 3 a** $H_0 : p = 0.4$ $H_1 : p > 0.4$

Critical region $X \geq 8$

$$\text{Power} = P(X \geq 8 | X \sim B(10, 0.5))$$

$$= 1 - P(X \leq 7)$$

$$= 1 - 0.9453 = 0.0547$$

- b** Power = $P(X \geq 8 | X \sim B(10, 0.8))$

Let $Y \sim B(10, 0.2)$ then

$$\text{Power} = P(X \geq 8 | X \sim B(10, 0.8)) = P(X \leq 2 | X \sim B(10, 0.2))$$

$$= 0.6778$$

- c** The test is more powerful for values of p further away from 0.4

- 4 a** $H_0 : p = \frac{1}{2}$ $H_1 : p < \frac{1}{2}$

Test A : critical region $X \leq 2$ where $X \sim B(10, p)$

$$\text{Size} = P(X \leq 2 | X \sim B(10, 0.5)) = 0.0547$$

4 b Power function = $P(X \leq 2 | X \sim B(10, p))$

$$\begin{aligned} &= \binom{10}{0} p^0 (1-p)^{10} + \binom{10}{1} p (1-p)^9 + \binom{10}{2} p^2 (1-p)^8 \\ &= (1-p)^{10} + 10p(1-p)^9 + \frac{10 \times 9}{2} p^2 (1-p)^8 \\ &= (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8 \end{aligned}$$

c Let the random variable Y denote the number of heads recorded in 5 spins of the coin, then $Y \sim B(5, p)$

Test B : $H_0 : p = \frac{1}{2}$ $H_1 : p < \frac{1}{2}$

Size = $P(\text{Type I error}) = P(H_0 \text{ rejected} | X \sim B(10, 0.5))$

= $P(\text{fails test 1}) + P(\text{passes test 1 then fails test 2})$

= $P(Y = 0 | X \sim B(10, 0.5)) + (1 - P(Y = 0 | X \sim B(10, 0.5)))P(Y = 0 | X \sim B(10, 0.5))$

= $0.03125 + (1 - 0.03125)0.03125$

= $0.03125 + 0.03027 = 0.0615$ (4 d.p.)

d Power function = $P(Y = 0 | X \sim B(5, p)) + ((1 - P(Y = 0 | X \sim B(5, p)))P(Y = 0 | X \sim B(5, p)))$
 $= (1-p)^5 + (1 - (1-p)^5)(1-p)^5$
 $= (1-p)^5(2 - (1-p)^5)$

e From the tables for the binomial cumulative distribution function

Power = $P(X \leq 2 | X \sim B(10, 0.25)) = 0.5256$

Power = $P(X \leq 2 | X \sim B(10, 0.35)) = 0.2616$

f Use test A as this is more powerful – the table shows test A has a higher power within the likely range of the parameter ($p < 0.5$).

5 a $H_0 : p = 0.15$ $H_1 : p < 0.15$

Assume H_0 , so that $X \sim \text{Geo}(0.15)$

Significance level 1%

Require $P(X \geq c) < 0.01$

So $(1 - 0.15)^{c-1} < 0.01$

$(c - 1) \log 0.85 < \log 0.01$

$$c - 1 > \frac{\log 0.01}{\log 0.85}$$

$c > 29.336$

So the critical value is 30 and the critical region is $X \geq 30$

Size = $P(H_0 \text{ rejected} | H_0 \text{ true}) = P(X \geq 30 | X \sim \text{Geo}(0.15))$

$$= (1 - 0.15)^{30-1} = 0.85^{29} = 0.0090 \text{ (4 d.p.)}$$

5 b Power function = $P(H_0 \text{ rejected} \mid X \sim \text{Geo}(p))$
 $= P(X \geq 30 \mid X \sim \text{Geo}(p)) = (1-p)^{29}$

6 a $H_0 : p = 0.7$

$H_1 : p \geq 0.7$

If 10 trials are done then under $B(10, 0.7)$

$P(X \geq 9) = 0.1493\dots$

$P(X \geq 10) = 0.02824\dots$

So the critical number of trials without a flat tyre is 10

Size of the test

$= P(\text{reject } H_0 \text{ when it is true})$

$= P(X \geq 10 \mid X \sim B(10, 0.7))$

$= 0.02824\dots$

≈ 0.028

b Power function of the test

$= P(\text{reject } H_0 \text{ when it is false}) = \lambda^{10}$

c $H_0 : p = 0.7$

$H_1 : p \geq 0.7$

If 12 trials are done then under $B(12, 0.7)$

$P(X \geq 11) = 0.085\dots$

$P(X \geq 12) = 0.013\dots$

So the critical number of trials without a flat tyre is 12

Power function of the test

$= P(\text{reject } H_0 \text{ when it is false}) = \lambda^{12}$

d Because $0.95^{10} > 0.95^{12}$
 the test is more powerful when 10 trials are done.