Probability generating functions 7B

1 a If $X \sim \mathrm{B}(n, p)$, then $\mathrm{G}_X(t) = (1 - p + pt)^n$

So $X \sim \mathrm{B}(4, 0.5)$, $\mathrm{G}_X(t) = (1 - 0.5 + 0.5t)^4 = (0.5 + 0.5t)^4 = 0.5^4(1+t)^4 = \frac{1}{16}(1+t)^4$

b $Y \sim \mathrm{B}(6, 0.2)$, $\mathrm{G}_Y(t) = (1 - 0.2 + 0.2t)^6 = (0.8 + 0.2t)^6$

c $X \sim \mathrm{B}(5, 0.9)$, $\mathrm{G}_X(t) = (1 - 0.9 + 0.9t)^5 = (0.1 + 0.9t)^5$

d If $X \sim \mathrm{Po}(\lambda)$, then $\mathrm{G}_X(t) = \mathrm{e}^{\lambda(t-1)}$

$X \sim \mathrm{Po}(3)$, $\mathrm{G}_X(t) = \mathrm{e}^{3(t-1)}$

e $X \sim \mathrm{Po}(1.7)$, $\mathrm{G}_X(t) = \mathrm{e}^{1.7(t-1)}$

f $Y \sim \mathrm{Po}(0.2)$, $\mathrm{G}_Y(t) = \mathrm{e}^{0.2(t-1)}$

2 a If $X \sim \mathrm{Geo}(p)$, then $\mathrm{G}_X(t) = \frac{pt}{1-(1-p)t}$

$X \sim \mathrm{Geo}(0.3)$, $\mathrm{G}_X(t) = \frac{0.3t}{1-(1-0.3)t} = \frac{0.3t}{1-0.7t} = \frac{3t}{10-7t}$

b $Y \sim \mathrm{Geo}(0.8)$, $\mathrm{G}_Y(t) = \frac{0.8t}{1-(1-0.8)t} = \frac{0.8t}{1-0.2t} = \frac{4t}{5-t}$

c If $X \sim$ Negative $\mathrm{B}(r, p)$, then $\mathrm{G}_X(t) = \left(\frac{pt}{1-(1-p)t}\right)^r$

$X \sim$ Negative $\mathrm{B}(3, 0.4)$, $\mathrm{G}_X(t) = \left(\frac{0.4t}{1-(1-0.4)t}\right)^3 = \left(\frac{0.4t}{1-0.6t}\right)^3 = \left(\frac{2t}{5-3t}\right)^3$

d $Y \sim$ Negative $\mathrm{B}(5, 0.9)$, $\mathrm{G}_Y(t) = \left(\frac{0.9t}{1-(1-0.9)t}\right)^5 = \left(\frac{0.9t}{1-0.1t}\right)^5 = \left(\frac{9t}{10-t}\right)^5$

3 a Let the random variable X be the number of sixes obtained in 5 rolls, then $X \sim \mathrm{B}(5, 0.2)$

So $\mathrm{G}_X(t) = (1 - 0.2 + 0.2t)^5 = (0.8 + 0.2t)^5$

b Let the random variable Y be the number of throws of the dice until a six is thrown, then $Y \sim \mathrm{Geo}(0.2)$

So $\mathrm{G}_Y(t) = \frac{0.2t}{1-(1-0.2)t} = \frac{0.2t}{1-0.8t} = \frac{t}{5-4t}$

3 **c** Let the random variable Y be the number of throws of the dice until two sixes have been thrown, then $Z \sim \text{Negative B}(2, 0.2)$

$$\text{So } \mathrm{G}_Z(t) = \left(\frac{0.2t}{1-(1-0.2)t}\right)^2 = \left(\frac{0.2t}{1-0.8t}\right)^2 = \left(\frac{t}{5-4t}\right)^2$$

4 **a** $X \sim \text{Po}(0.3)$

b $\text{P}(X=1) = \dfrac{\mathrm{e}^{-0.3}0.3^1}{1!} = 0.3\mathrm{e}^{-0.3} = 0.2222 \text{ (4 d.p.)}$

c If $X \sim \text{Po}(\lambda)$, then $\mathrm{G}_X(t) = \mathrm{e}^{\lambda(t-1)}$

So $\mathrm{G}_X(t) = \mathrm{e}^{0.3(t-1)}$

5 **a** $X \sim \text{Geo}(0.35)$

b $\text{P}(X=6) = 0.35(1-0.35)^5 = 0.35 \times 0.65^5 = 0.0406 \text{ (4 d.p.)}$

c $\mathrm{G}_X(t) = \dfrac{0.35t}{1-(1-0.35)t} = \dfrac{0.35t}{1-0.65t} = \dfrac{7t}{20-13t}$

6 $X \sim \text{B}(4, 0.8)$, so $\text{P}(X=x) = \dbinom{4}{x}0.8^x(1-0.8)^{4-x}$

$$\begin{aligned}\mathrm{G}_X(t) &= \sum t^x \text{P}(X=x)\\ &= (0.2)^4 + 4(0.8)(0.2)^3t + 6(0.8)^2(0.2)^2t^2 + 4(0.8)^3(0.2)t^3 + (0.8)^4t^4\\ &= (0.2)^4 + 4(0.2)^3(0.8t) + 6(0.2)^2(0.8t)^2 + 4(0.2)(0.8t)^3 + (0.8t)^4\\ &= (0.2+0.8t)^4\end{aligned}$$

7 $X \sim \text{Po}(3.5)$, so $\text{P}(X=x) = \dfrac{\mathrm{e}^{-3.5}3.5^x}{x!}$

$$\begin{aligned}\mathrm{G}_X(t) &= \sum t^x \text{P}(X=x) = \sum t^x \frac{\mathrm{e}^{-3.5}3.5^x}{x!}\\ &= \mathrm{e}^{-3.5}\sum \frac{t^x 3.5^x}{x!} = \mathrm{e}^{-3.5}\sum \frac{(3.5t)^x}{x!}\\ &= \mathrm{e}^{-3.5}\left(1 + 3.5t + \frac{(3.5t)^2}{2!} + \frac{(3.5t)^3}{3!} + \ldots\right)\end{aligned}$$

The expression in brackets is the Maclaurin expansion of e^x where $x = 3.5t$, so:

$\mathrm{G}_X(t) = \mathrm{e}^{-3.5}\mathrm{e}^{3.5t} = \mathrm{e}^{3.5t-3.5} = \mathrm{e}^{3.5(t-1)}$

8 $Y \sim \text{Geo}(0.7)$, so $P(Y = y) = (1-0.7)^{y-1}0.7$

$$G_Y(t) = \sum_{y=1}^{\infty} t^y P(Y = y) = \sum_{y=1}^{\infty} t^y 0.3^{y-1} 0.7$$

$$= 0.7t\sum_{y=1}^{\infty}(0.3t)^{y-1} = 0.7t\sum_{y=0}^{\infty}(0.3t)^y$$

$$= 0.7t(1+0.3t+(0.3t)^2+\dots)$$

The bracketed expression is the sum of an infinite geometric series, with first term 1 and common ratio $0.3t$. Using the formula for the sum of a geometric series gives:

$$(1+0.3t+(0.3t)^2+\dots) = \frac{1}{1-0.3t}$$

So $G_Y(t) = 0.7t\frac{1}{1-0.3t} = \frac{0.7t}{1-0.3t}$

9 $X \sim B(n, p)$, so $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$G_X(t) = \sum t^x P(X = x) = \sum t^x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum \binom{n}{x} (pt)^x (1-p)^{n-x}$$

$$= (1-p)^n + \binom{n}{1}(1-p)^{n-1}pt + \binom{n}{2}(1-p)^{n-2}(pt)^2 + \binom{n}{3}(1-p)^{n-3}(pt)^3 + \dots + (pt)^n$$

This is the binomial expansion of $(a+b)^n$ (see Pure Year 1, Chapter 8) with $a = 1-p$ and $b = pt$

So $G_X(t) = (1-p+pt)^n$

10 $X \sim \text{Po}(\lambda)$, so $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$$G_X(t) = \sum t^x P(X = x) = \sum t^x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum\frac{(\lambda t)^x}{x!} = e^{-\lambda}\left(1+\lambda t+\frac{(\lambda t)^2}{2!}+\frac{(\lambda t)^3}{3!}+\dots\right)$$

The bracketed expression is the Maclaurin expansion of e^x where $x = \lambda t$

So $G_X(t) = e^{-\lambda}e^{\lambda t} = e^{\lambda t - \lambda} = e^{\lambda(t-1)}$

11 $Y \sim \text{Geo}(p)$, so $\text{P}(Y = y) = p(1-p)^{y-1}$

$$\text{G}_Y(t) = \sum t^y \text{P}(Y = y) = \sum_{y=1}^{\infty} t^y p(1-p)^{y-1}$$

$$= pt\sum_{y=1}^{\infty}\left(t(1-p)\right)^{y-1} = pt\sum_{y=0}^{\infty}\left(t(1-p)\right)^{y}$$

$$= pt\left(1+(1-p)t+((1-p)t)^2+\ldots\right)$$

The bracketed expression is the sum of an infinite geometric series, with first term 1 and common ratio $(1-p)t$. Using the formula for the sum of a geometric series gives:

So $\text{G}_Y(t) = pt\dfrac{1}{1-(1-p)t} = \dfrac{pt}{1-(1-p)t}$