Geometric and negative binomial distributions 3C

1 Let X denote the number of throws required to throw a 4 for the third time.
$X \sim \text{Negative B}(3, 0.25)$

$$P(X = 6) = \binom{5}{2} \times (0.25)^3 \times (0.75)^3 = 0.0659 \text{ (4 d.p.)}$$

2 Let X denote the number of spins required to get four heads.
$X \sim \text{Negative B}(4, 0.55)$

$$P(X = 7) = \binom{6}{3} \times (0.55)^4 \times (0.45)^3 = 0.1668 \text{ (4 d.p.)}$$

3 Let X denote the number of shots required to hit the bullseye for the second time.
$X \sim \text{Negative B}(2, 0.15)$

$$P(X = 10) = \binom{9}{1} \times (0.15)^2 \times (0.85)^8 = 0.0552 \text{ (4 d.p.)}$$

4 **a** To pick her first correct answer on her third question, Denise's answers in sequence would be wrong, wrong, correct. Let X be the event that this sequence of answers occurs.

$$P(X) = P(\text{wrong answer}) \times P(\text{wrong answer}) \times P(\text{correct answer})$$
$$= 0.75 \times 0.75 \times 0.25 = 0.1406 \text{ (4 d.p.)}$$

b Let Y denote the number of questions required to get four correct answers.
$Y \sim \text{Negative B}(4, 0.25)$

$$P(Y = 7) = \binom{6}{3} \times (0.25)^4 \times (0.75)^3 = 0.0330 \text{ (4 d.p.)}$$

c Let Z be the number of correct answers that Denise gets in the first 10 questions.
$Z \sim B(10, 0.25)$

$$P(Z = 2) = \binom{10}{2} \times (0.25)^2 \times (0.75)^8 = 0.2816 \text{ (4 d.p.)}$$

d $$P(Z \geqslant 3) = 1 - P(Z \leqslant 2)$$
$$= 1 - 0.5256 = 0.4744 \text{ (4 d.p.)}$$

5 **a** To win his first match on the fourth attempt, Eliot's in sequence would be lose, lose, lose, win. Let X be the event that this sequence of answers occurs.

$$P(X) = P(\text{lose}) \times P(\text{lose}) \times P(\text{lose}) \times P(\text{win})$$
$$= (0.7)^3(0.3) = 0.1029$$

b Let Y be the number of matches that Eliot wins in 10 matches, so $Y \sim B(10, 0.3)$
$P(Y \leqslant 3) = 0.6496$ (4 d.p.)

c Let Z denote the number of matches Eliot takes to get three wins. So $Z \sim \text{Negative B}(3, 0.3)$

$$P(Z = 8) = \binom{7}{2} \times (0.3)^3 \times (0.7)^5 = 0.0953 \text{ (4 d.p.)}$$

d Let A be the number of matches that Eliot wins in 12 matches, so $A \sim B(12, 0.3)$

$$P(A \geqslant 5) = 1 - P(A \leqslant 4)$$
$$= 1 - 0.7237 = 0.2763 \text{ (4 d.p.)}$$

6 **a** Let X denote the number of games played before Francesca wins four times.
$X \sim \text{Negative B}(4, 0.55)$

$$\text{P}(X=6)=\binom{5}{3}\times(0.55)^4\times(0.45)^2=0.1853 \text{ (4 d.p.)}$$

b The games are independent and the probability of Francesca winning is the same in any game.

c This requires finding the probability that it takes Francesca four more games to win twice after winning her first game, so that she wins her third game on the fifth attempt.
Let Y denote the number of games played before Francesca wins two games. So $Y \sim \text{Negative B}(2, 0.55)$ and find $\text{P}(Y=4)$

$$\text{P}(Y=4)=\binom{3}{1}\times(0.55)^2\times(0.45)^2=0.1838 \text{ (4 d.p.)}$$

d Let A be the number of games that Francesca wins in 10 matches, so $A \sim \text{B}(10, 0.55)$

$$\begin{aligned}\text{P}(A \geqslant 7) &= 1-\text{P}(A \leqslant 6)\\ &= 1-0.7340=0.2660 \text{ (4 d.p.)}\end{aligned}$$

7 **a** Let X denote the number of trials it takes before Gerald cures seven patients.
$X \sim \text{Negative B}(7, 0.8)$

$$\text{P}(X=10)=\binom{9}{6}\times(0.8)^7\times(0.2)^3=0.1409 \text{ (4 d.p.)}$$

b The trials are independent and the probability of curing a patient is the same in each trial.

c Let Y be the number of patients that Gerald cures in 12 trials, so $Y \sim \text{B}(12, 0.8)$
$\text{P}(Y \geqslant 7)=1-\text{P}(Y \leqslant 6)=1-0.0194=0.9806$ (4 d.p.)

d Let Z be the number of patients that Gerald cures in 20 trials, so $Z \sim \text{B}(20, 0.8)$
$P(Z \leqslant 14)=0.1958$ (4 d.p.)

8 **a** Let X denote the number of experiments Harriet conducts before she is successful twice.
$X \sim \text{Negative B}(2, 0.1)$

$$\text{P}(X=20)=\binom{19}{1}\times(0.1)^2\times(0.9)^{18}=0.0285 \text{ (4 d.p.)}$$

b Let Y be the number of successful experiments that Harriet conducts in 30 attempts, so $Y \sim \text{B}(30, 0.1)$
$\text{P}(Y \leqslant 3)=0.6474$ (4 d.p.)

c This is equivalent to finding the probability that it takes 25 attempts (30 – 5 = 25) to conduct three more successful experiments. Let Z denote the number of experiments Harriet conducts before she is successful three times.
$Z \sim \text{Negative B}(25, 0.1)$

$$\text{P}(Z=25)=\binom{24}{2}\times(0.1)^3\times(0.9)^{22}=0.0272 \text{ (4 d.p.)}$$

d Let A be the number of successful experiments that Harriet conducts in 25 attempts, so $A \sim \text{B}(25, 0.1)$

$$\begin{aligned}\text{P}(A \geqslant 3) &= 1-\text{P}(A \leqslant 2)\\ &= 1-0.5371=0.4629 \text{ (4 d.p.)}\end{aligned}$$

9 a $\mathrm{P}(X=10)=\binom{9}{4}\times(0.7)^3\times(0.3)^5=0.0515$ (4 d.p.)

b $\mathrm{P}(X=5)+\mathrm{P}(X=6)=(0.7)^5+\binom{5}{4}\times(0.7)^5\times0.3=0.16807+0.25211=0.4202$ (4 d.p.)

c This requires calculating the probability that the fifth success occurs on or before the 15th trial. Let Y be the number of successes in 15 attempts, so $Y\sim\mathrm{B}(15,0.7)$

$\mathrm{P}(X\leqslant15)=\mathrm{P}(Y\geqslant5)$

$\mathrm{P}(Y\geqslant5)=1-\mathrm{P}(Y\leqslant4)=1-0.007=0.9993$ (4 d.p.)

d This requires calculating the probability that the fifth success occurs after the 12th trial. Let Z be the number of successes in 12 attempts, so $Z\sim\mathrm{B}(12,0.7)$

$\mathrm{P}(X>12)=\mathrm{P}(Z\leqslant4)=0.0095$ (4 d.p.)

10 a $D\sim$ Negative $\mathrm{B}(3,p)$

b i $\mathrm{P}(X=7)=\binom{6}{2}\times(0.35)^3\times(0.75)^4=0.1148$ (4 d.p.)

ii Let E be the number of bulleyes that the darts player scores in 7 throws, so $E\sim\mathrm{B}(7,0.35)$
$\mathrm{P}(D\geqslant8)=\mathrm{P}(E\leqslant2)=0.5323$ (4 d.p.)

iii This requires finding the probability that it takes 8 throws after the first throw to score two more bullseyes. Let F denote the number of throws required to score 2 bullseyes.
$F\sim$ Negative $\mathrm{B}(2,0.35)$

$\mathrm{P}(F=8)=\binom{7}{1}\times(0.35)^2\times(0.65)^6=0.0647$ (4 d.p.)

c The probability of hitting the bullseye might not be constant. The darts player may get better as she gets more practice, so the probability of scoring a bullseye might increase.

Challenge

a $\mathrm{P}(Y\leqslant8)=1-\mathrm{P}(Y>8)$

As $Y\sim$ Negative $\mathrm{B}(3,0.4)$, $\mathrm{P}(Y>8)$ is the probability of 2 or fewer successes in the first 8 trials.

Let X be the number of successes in 8 trials, $X\sim\mathrm{B}(8,0.4)$

So $\mathrm{P}(Y>8)=\mathrm{P}(X\leqslant2)$

Therefore $\mathrm{P}(Y\leqslant8)=1-\mathrm{P}(X\leqslant2)$

This gives $\mathrm{P}(Y\leqslant8)=1-\mathrm{F}_{8,0.4}(2)$

b $\mathrm{P}(Y\leqslant y)=1-\mathrm{P}(Y>y)$

As $Y\sim$ Negative $\mathrm{B}(r,p)$, $\mathrm{P}(Y>y)$ is the probability of $(r-1)$ or fewer successes in the first y trials.

Let X be the number of successes in y trials, $X\sim\mathrm{B}(y,p)$

So $\mathrm{P}(Y>y)=\mathrm{P}(X\leqslant r-1)$

Therefore $\mathrm{P}(Y\leqslant y)=1-\mathrm{P}(X\leqslant r-1)$

This gives $\mathrm{P}(Y\leqslant y)=1-\mathrm{F}_{y,p}(r-1)$