

### Discrete random variables 1A

1 a The probability distribution for  $X^2$  is:

$x$	2	4	6	8
$P(X = x)$	0.3	0.3	0.2	0.2
$x^2$	4	16	36	64
$P(X^2 = x^2)$	0.3	0.3	0.2	0.2

Note that for this variable  $P(X = x) = P(X^2 = x^2)$  as  $X$  only takes positive values.

$$E(X) = \sum xP(X = x)$$

$$= 2 \times 0.3 + 4 \times 0.3 + 6 \times 0.2 + 8 \times 0.2 = 4.6$$

$$E(X^2) = \sum x^2P(X = x)$$

$$= 4 \times 0.3 + 16 \times 0.3 + 36 \times 0.2 + 64 \times 0.2 = 26$$

b The probability distribution for  $X$  is:

$x$	-2	-1	1	2
$P(X = x)$	0.1	0.4	0.1	0.4
$x^2$	4	1	1	4

In this case,  $X$  can take negative values, so calculate the values of  $P(X^2 = x^2)$ .

$$P(X^2 = 1) = P(X = -1) + P(X = 1) = 0.4 + 0.1 = 0.5$$

$$P(X^2 = 4) = P(X = -2) + P(X = 2) = 0.1 + 0.4 = 0.5$$

The probability distribution for  $X^2$  is:

$x^2$	1	4
$P(X^2 = x^2)$	0.5	0.5

$$E(X) = \sum xP(X = x)$$

$$= -2 \times 0.1 + (-1) \times 0.4 + 1 \times 0.1 + 2 \times 0.4$$

$$= 0.3$$

$$E(X^2) = \sum x^2P(X = x)$$

$$= 4 \times 0.1 + 1 \times 0.4 + 1 \times 0.1 + 4 \times 0.4$$

$$= 2.5$$

Using the  $X^2$  distribution to calculate  $E(X^2)$  gives the same result

$$E(X^2) = \sum x^2P(X^2 = x^2) = 1 \times 0.5 + 4 \times 0.5 = 2.5$$

$$\begin{aligned}
 2 \quad E(X) &= \sum x P(X = x) \\
 &= (1 \times 0.1) + (2 \times 0.1) + (3 \times 0.1) + (4 \times 0.2) + (5 \times 0.4) + (6 \times 0.1) \\
 &= 0.1 + 0.2 + 0.3 + 0.8 + 2.0 + 0.6 \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \sum x^2 P(X = x) \\
 &= (1 \times 0.1) + (4 \times 0.1) + (9 \times 0.1) + (16 \times 0.2) + (25 \times 0.4) + (36 \times 0.1) \\
 &= 0.1 + 0.4 + 0.9 + 3.2 + 10 + 3.6 \\
 &= 18.2
 \end{aligned}$$

3 a The probability distribution for  $X$  is:

$x$	2	3	6
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

The probability distribution for  $X^2$  is:

$x^2$	4	9	36
$P(X^2 = x^2)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$\begin{aligned}
 b \quad E(X) &= \sum x P(X = x) \\
 &= 2 \times \frac{1}{2} + 3 \times \frac{1}{3} + 6 \times \frac{1}{6} \\
 &= 1 + 1 + 1 \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \sum x^2 P(X = x) \\
 &= 4 \times \frac{1}{2} + 9 \times \frac{1}{3} + 36 \times \frac{1}{6} \\
 &= 11
 \end{aligned}$$

c  $(E(X))^2 = 3^2 = 9$  and  $E(X^2) = 11$  from part b  
 So  $(E(X))^2$  does not equal  $E(X^2)$

4 a The probability distribution for  $X$  is:

$x$	1	2	3	4	5
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

b  $E(X) = \sum xP(X=x)$

$$= 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{16} + 5 \times \frac{1}{16}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{6}{16} + \frac{9}{16} = \frac{31}{16}$$

$$= 1.9375$$

$E(X^2) = \sum x^2P(X=x)$

$$= 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4} + 3^2 \times \frac{1}{8} + 4^2 \times \frac{1}{16} + 5^2 \times \frac{1}{16}$$

$$= 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 9 \times \frac{1}{8} + 16 \times \frac{1}{16} + 25 \times \frac{1}{16}$$

$$= 5\frac{3}{16} = \frac{83}{16}$$

$$= 5.1875$$

c  $(E(X))^2 = (1.9375)^2 = 3.7539$  (4 d.p.)

So  $(E(X))^2$  does not equal  $E(X^2)$

5 The probabilities add up to 1, so

$$0.1 + a + b + 0.2 + 0.1 = 1$$

$$a + b = 0.6 \quad (1)$$

$E(X) = \sum xP(X=x) = 2.9$ , so

$$(1 \times 0.1) + (2 \times a) + (3 \times b) + (4 \times 0.2) + (5 \times 0.1) = 2.9$$

$$0.1 + 2a + 3b + 0.8 + 0.5 = 2.9$$

$$2a + 3b = 1.5 \quad (2)$$

Multiply (1) by 2

$$2a + 2b = 1.2 \quad (3)$$

Subtract equation (3) from (1) to give

$$b = 0.3$$

Substitute the value of  $b$  in equation (3) to obtain

$$2a + 0.6 = 1.2$$

$$\Rightarrow a = 0.3$$

$$\begin{aligned}6 \quad E(X) &= \sum xP(X=x) = 0.3, \text{ so} \\ 0.3 &= (-2 \times 0.1) + (-1 \times a) + (1 \times b) + (2 \times c) \\ 0.5 &= -a + b + 2c \quad (1)\end{aligned}$$

$$\begin{aligned}E(X^2) &= \sum x^2P(X=x), \text{ so} \\ 1.9 &= (4 \times 0.1) + (1 \times a) + (1 \times b) + (4 \times c) \\ 1.5 &= a + b + 4c \quad (2)\end{aligned}$$

$$\begin{aligned}\sum P(X=x) &= 1, \text{ so} \\ 1 &= 0.1 + a + b + c \\ 0.9 &= a + b + c \quad (3)\end{aligned}$$

$$\begin{aligned}(2) - (3) &\Rightarrow 0.6 = 3c \\ & \quad c = 0.2\end{aligned}$$

$$\begin{aligned}(3) - (1) &\Rightarrow 0.4 = 2a - c \\ 2a &= 0.4 + c = 0.6 \\ a &= 0.3\end{aligned}$$

Substituting for  $a$  in (3) gives

$$\begin{aligned}0.9 &= 0.3 + b + 0.2 \\ 0.4 &= b\end{aligned}$$

So the full solution is

$$a = 0.3, \quad b = 0.4, \quad c = 0.2$$

7 The probability distribution for  $X$  is:

$x$	-2	-1	0	5
$P(X = x)$	$3a$	$2a$	$a$	$b$

$$E(X) = \sum xP(X = x) = 1.2, \text{ so}$$

$$1.2 = -2 \times 3a - 1 \times 2a + 0 \times a + 5 \times b$$

$$1.2 = -6a - 2a + 5b$$

$$1.2 = -8a + 5b \quad (1)$$

$$\sum P(X = x) = 1, \text{ so}$$

$$1 = 3a + 2a + a + b$$

$$1 = 6a + b \quad (2)$$

$$(2) \times 5 \Rightarrow 5 = 30a + 5b \quad (3)$$

$$(3) - (1) \Rightarrow 3.8 = 38a \Rightarrow a = 0.1$$

Substituting for  $a$  in equation (2) gives

$$b = 1 - 6a = 1 - 0.6 = 0.4$$

So the full solution is

$$a = 0.1, \quad b = 0.4$$

8 a Suppose the probability distribution for  $X$  is:

$x$	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$a$	$b$

$$E(X) = \sum xP(X = x) = 4.1, \text{ so}$$

$$4.1 = 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} + 4 \times \frac{1}{8} + 5 \times a + 6 \times b$$

$$4.1 = \frac{10}{8} + 5a + 6b$$

$$2.85 = 5a + 6b \quad (1)$$

$$\sum P(X = x) = 1, \text{ so}$$

$$1 = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + a + b$$

$$0.5 = a + b \quad (2)$$

$$(2) \times 5 \Rightarrow 2.5 = 5a + 5b \quad (3)$$

$$(1) - (3) \Rightarrow 0.35 = b$$

Substituting for  $b$  in equation (2) gives

$$0.5 = a + 0.35 \Rightarrow a = 0.15$$

So the full solution is

$$a = 0.15 = \frac{3}{20}, \quad b = 0.35 = \frac{7}{20}$$

So the full probability distribution for  $X$  is:

$x$	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{20}$	$\frac{7}{20}$

8 b Let  $X$  = number of 6s in 10 rolls of the dice, then  $X \sim B(10, \frac{7}{20})$

To answer the question requires finding  $P(X \geq 3)$

$$P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$$

Use the formula for the probability mass function of a binomial distribution

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$$

In this case  $n = 10$  and  $p = 0.35$

$$\begin{aligned} P(X \geq 3) &= 1 - \left( \binom{10}{0} \times 0.35^0 \times 0.65^{10} + \binom{10}{1} \times 0.35^1 \times 0.65^9 + \binom{10}{2} \times 0.35^2 \times 0.65^8 \right) \\ &= 1 - (1 \times 1 \times 0.013463 + 10 \times 0.35 \times 0.020712 + 45 \times 0.1225 \times 0.031864) \\ &= 1 - (0.013463 + 0.072492 + 0.17565) = 1 - 0.261605 \\ &= 0.7384 \text{ (4 d.p.)} \end{aligned}$$

9 This is the result of each throw of the dice:

<b>Score</b>	1	2	3	4	5	6
<b>Probability</b>	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
<b>Prize (£)</b>	0	0	0	$P$	$P$	5

Let  $X$  be the amount paid out in prize money. So, using data from the table above, the probability distribution for  $X$  is:

<b><math>x</math></b>	0	$P$	5
<b><math>P(X = x)</math></b>	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$E(X) = \sum xP(X = x)$  is the expected amount paid out in prize money.

$$E(X) = 0 \times \frac{1}{2} + P \times \frac{1}{3} + 5 \times \frac{1}{6}$$

Let expected profit = 0

Expected profit = cost of game – expected amount paid out in prize money

$$0 = 1 - \left( \frac{1}{3}P + \frac{5}{6} \right)$$

$$\Rightarrow \frac{1}{3}P = \frac{1}{6} \Rightarrow P = 0.5$$

So if Jorge awards a prize of £0.50 for players throwing 4 or 5 he will break even with his game and not make a loss. The maximum value of  $P$  is £0.50.

## Challenge

When three dice are thrown there are  $6 \times 6 \times 6 = 216$  outcomes.

There is only one way of the number 1 being the highest score on the three dice and that is 1, 1, 1.

To achieve the highest score of 2, each dice must be either 1 or 2. So there are  $2 \times 2 \times 2 = 8$  ways for the highest score on three dice to be no more than 2. But one of those is 1, 1, 1, which gives a highest score of 1 so this needs to be subtracted to leave 7 possible ways for a highest score of 2.

To achieve the highest score of 3, each dice must be either 1 or 2 or 3. So there are  $3 \times 3 \times 3 = 27$  ways for the highest score on three dice to be no more than 3. But one of those is 1, 1, 1, which gives a highest score of 1 and there are 7 possible ways for a highest score of 2 so these both need to be subtracted to give 19 ways of getting a highest score of 3.

Using this approach, this is the number of ways of getting each highest score:

Highest score on the three dice	Working	Number of ways
1	1, 1, 1	1
2	$2 \times 2 \times 2 - 1$	7
3	$3 \times 3 \times 3 - 7 - 1$	19
4	$4 \times 4 \times 4 - 19 - 7 - 1$	37
5	$5 \times 5 \times 5 - 37 - 19 - 7 - 1$	61
6	$6 \times 6 \times 6 - 61 - 37 - 19 - 7 - 1$	91

Converting the number of ways into probabilities gives:

$x$	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{216}$	$\frac{7}{216}$	$\frac{19}{216}$	$\frac{37}{216}$	$\frac{61}{216}$	$\frac{91}{216}$

$$E(X) = \sum xP(X = x)$$

$$\begin{aligned}
 &= 1 \times \frac{1}{216} + 2 \times \frac{7}{216} + 3 \times \frac{19}{216} + 4 \times \frac{37}{216} + 5 \times \frac{61}{216} + 6 \times \frac{91}{216} \\
 &= \frac{1071}{216} = \frac{119}{24} = 4.9583 \text{ (4 d.p.)}
 \end{aligned}$$