Number theory 1B

1 a $7 \mid 7 \Rightarrow \gcd(7,7) = 7$

b $100 = 5 \times 20 \Rightarrow \gcd(100,20) = 20$

c $15 = 3 \times 5$ and $18 = 2 \times 3^2 \Rightarrow \gcd(15,18) = 3$

2 If $\gcd(p,42) = 6$ and p is a non-negative integer, then p can be any multiple of 6 that is not divisible by 42, for instance p can be 6, 12, 18, …

3 a $78 = 2 \times 32 + 14$
$32 = 2 \times 14 + 4$
$14 = 3 \times 4 + 2$
$4 = 2 \times 2 + 0$
So $\gcd(32,78) = 2$

b $104 = 1 \times 91 + 13$
$91 = 7 \times 13 + 0$
So $\gcd(91,104) = 13$

c $172 = 2 \times 64 + 44$
$64 = 1 \times 44 + 20$
$44 = 2 \times 20 + 4$
$20 = 5 \times 4 + 0$
So $\gcd(172,64) = 4$

d $167 = 1 \times 117 + 50$
$117 = 2 \times 50 + 17$
$50 = 2 \times 17 + 16$
$17 = 1 \times 16 + 1$
$16 = 1 \times 16 + 0$
So $\gcd(167,117) = 1$

e $-323 = -2 \times 221 + 119$
$221 = 1 \times 119 + 102$
$119 = 1 \times 102 + 17$
$102 = 6 \times 17 + 0$
So $\gcd(-323,221) = 17$

f $1292 = 1 \times 884 + 408$
$884 = 2 \times 408 + 68$
$408 = 6 \times 68 + 0$
So $\gcd(1292,884) = 68$

4 $910 = 6 \times 143 + 52$
$143 = 2 \times 52 + 39$
$52 = 1 \times 39 + 13$
$39 = 3 \times 13 + 0$
So highest common factor is 13

5 **a** $1050 = 4 \times 222 + 162$
$222 = 1 \times 162 + 60$
$162 = 2 \times 60 + 42$
$60 = 1 \times 42 + 18$
$42 = 2 \times 18 + 6$
$18 = 3 \times 6 + 0$
So $\gcd(222, 1050) = 6$

b $\frac{222}{1050} = \frac{6 \times 37}{6 \times 175} = \frac{37}{175}$

6 **a** From question **3a**, gcd(32, 78) = 2. Working backwards through the steps of the Euclidean algorithm from the solution to question **3a** gives:

$$\begin{aligned} 2 &= 14 - 3(4) \\ &= 14 - 3(32 - (2(14)) \\ &= 7(14) - 3(32) \\ &= 7(78 - 2(32)) - 3(32) \\ &= -17(32) + 7(78) \end{aligned}$$

So $x = -17$, $y = 7$

Note that $x = 22$, $y = -9$ also satisfies $32x + 78y = 2$

b From question **3b**, gcd(91, 104) = 13. Working backwards through the steps of the Euclidean algorithm from the solution to question **3b** gives:

$$\begin{aligned} 13 &= 104 - 1(91) \\ &= -1(91) + 1(104) \end{aligned}$$

So $x = -1$, $y = 1$

6 c $12\,378 = 4 \times 3054 + 162$

$3054 = 18 \times 162 + 138$

$162 = 1 \times 138 + 24$

$138 = 5 \times 24 + 18$

$24 = 1 \times 18 + 6$

$18 = 3 \times 6 + 0$

So $\gcd(12\,378, 3054) = 6$

Working backwards through the steps of the Euclidean algorithm gives:

$$\begin{aligned} 6 &= 24 - 1(18) \\ &= 24 - (138 - 5(24)) \\ &= 6(24) - 1(138) \\ &= 6(162 - 1(138)) - 1(138) \\ &= 6(162) - 7(138) \\ &= 6(162) - 7(3054 - 18(162)) \\ &= 132(162) - 7(3054) \\ &= 132(12\,378 - 4(3054)) - 7(3054) \\ &= 132(12\,378) - 535(3054) \end{aligned}$$

So $x = 132$, $y = -535$

d $272 = -2 \times (-119) + 34$

$-119 = -4 \times 34 + 17$

$34 = 2 \times 17 + 0$

So $\gcd(-119, 272) = 17$

Working backwards through the steps of the Euclidean algorithm gives:

$$\begin{aligned} 16 &= -119 + 4(34) \\ &= -119 + 4(272 + 2(-119)) \\ &= 9(-119) + 4(272) \end{aligned}$$

So $x = 9$, $y = 4$

6 e $2378 = 1 \times 1769 + 609$

$1769 = 2 \times 609 + 551$

$609 = 1 \times 551 + 58$

$551 = 9 \times 58 + 29$

$58 = 2 \times 29 + 0$

So $\gcd(2378, 1769) = 29$

Working backwards through the steps of the Euclidean algorithm gives:

$$\begin{aligned}29 &= 551 - 9(58)\\ &= 551 - 9(609 - 551)\\ &= -9(609) + 10(551)\\ &= -9(609) + 10(1769 - 2(609))\\ &= -29(609) + 10(1769)\\ &= -29(2378 - 1769) + 10(1769)\\ &= -29(2378) + 39(1769)\end{aligned}$$

So $x = -29$, $y = 39$

f $2581 = -1 \times (-2059) + 522$

$-2059 = -4 \times 522 + 29$

$522 = 18 \times 29 + 0$

So $\gcd(-2059, 2581) = 29$

Working backwards through the steps of the Euclidean algorithm gives:

$$\begin{aligned}29 &= 1(-2059) + 4(522)\\ &= 1(-2059) + 4(2581 + 1(-2059))\\ &= 5(-2059) + 4(2581)\end{aligned}$$

So $x = 5$, $y = 4$

7 a $39 = 2 \times 16 + 7$

$16 = 2 \times 7 + 2$

$7 = 3 \times 2 + 1$

So $\gcd(39, 16) = 1$, hence 39 and 16 are relatively prime.

b Working backwards:

$$\begin{aligned}1 &= 1(7) - 3(2)\\ &= 1(7) - 3(16 - 2(7))\\ &= 7(7) - 3(16)\\ &= 7(39 - 2(16)) - 3(16)\\ &= 7(39) - 17(16)\end{aligned}$$

So $p = 7$, $q = -17$

8 $170 = 8 \times 21 + 2$
$21 = 10 \times 2 + 1$
So $\gcd(170, 21) = 1$

Working backwards:
$1 = 1(21) - 10(2)$
$= 1(21) - 10(170 - 8(21))$
$= -10(170) + 81(21)$
So $a = -10$, $b = 81$

9 **a** $172 = 8 \times 20 + 12$
$20 = 1 \times 12 + 8$
$12 = 1 \times 8 + 4$
$8 = 2 \times 4 + 0$
So $\gcd(172, 20) = 4$

Working backwards:
$4 = 1(12) - 1(8)$
$= 1(12) - (20 - 12)$
$= 2(12) - 1(20)$
$= 2(172 - 8(20)) - (20)$
$= 2(172) - 17(20)$
So $x = 2$, $y = -17$

b $2 \times 172 - 17 \times 20 = 4 \Rightarrow 2 \times 172 \times 25 - 17 \times 20 \times 25 = 4 \times 25 \Rightarrow (50)172 + (-425)20 = 100$
Therefore solutions to $172x + 20y = 100$ are $x = 50$, $y = -425$.

10 Dividing both sides by 3 gives $33a + 115b = 100$
Finding the greatest common divisor of 33 and 155:
$115 = 3 \times 33 + 16$
$33 = 2 \times 16 + 1$
$16 = 16 \times 1 + 0$
So $\gcd(33, 115) = 1$

Working backwards:
$1 = 33 - 2(16)$
$= 33 - 2(115 - 3(33))$
$= 7(33) - 2(115)$
Hence $7 \times 33 - 2 \times 115 = 1 \Rightarrow 700 \times 33 - 200 \times 115 = 100$, and so $a = 700$, $b = -200$.

11 a $f(1) = 11,\ g(1) = 7,$ hence $\gcd(f(1), g(1)) = 1$ because the numbers 11 and 7 are both prime.

b $8n + 3 = 1 \times (5n + 2) + (3n + 1)$
$5n + 2 = 1 \times (3n + 1) + (2n + 1)$
$3n + 1 = 1 \times (2n + 1) + n$
$2n + 1 = 2 \times (n) + 1$
So $\gcd(8n + 3, 5n + 2) = \gcd(f(n), g(n)) = 1$
So f(n) and g(n) are relatively prime for all $n \in \mathbb{Z}^+$

12 a Let $\gcd(a, a + x) = d$, then there exist $m, n \in \mathbb{Z}$ such that $a = md$ and $a + x = nd$
So $x = (n - m)d$. As $n - m \in \mathbb{Z}$, $d \mid x$ so $\gcd(a, a + x) \mid x$.

b let $x = 1$, then from part **a** $\gcd(a, a+1) \mid 1$. As 1 is only divisible by 1, this gives $\gcd(a, a+1) = 1$. Therefore any two consecutive integers are relatively prime.

13 a $63 = -2 \times (-23) + 17$
$-23 = -2 \times 17 + 11$
$17 = 1 \times 11 + 6$
$11 = 1 \times 6 + 5$
$6 = 1 \times 5 + 1$
So $\gcd(63, -23) = 1$

b Working backwards:
$$\begin{aligned}
1 &= 6 - 5 \\
&= 6 - (11 - 6) \\
&= 2(6) - 11 \\
&= 2(17 - 11) - 11 \\
&= 2(17) - 3(11) \\
&= 2(17) - 3(-23 + 2(17)) \\
&= -4(17) - 3(-23) \\
&= -4(63 + 2(-23)) - 3(-23) \\
&= -4(63) - 11(-23)
\end{aligned}$$
Hence $-4 \times 63 + 11 \times 23 = 1 \Rightarrow 28 \times 63 - 77 \times 23 = -7$, and so $x_0 = 28$, $y_0 = 77$.

c
$$\begin{aligned}
63x - 23y &= 63(x_0 - 23t) - 23(y_0 - 63t) \\
&= 63(28 - 23t) - 23(77 - 63t) \\
&= 28 \times 63 - 77 \times 23 - (63 \times 23)t + (63 \times 23)t \\
&= 28 \times 63 - 77 \times 23 = -7 \qquad \text{(from part } \mathbf{a}\text{)}
\end{aligned}$$
So $x = 28 - 23t$ and $y = 77 - 63t$ is a solution to $63x - 23y = -7$ for any $t \in \mathbb{Z}$

d $x = 28 - 23t$ and $y = 77 - 63t$.
As t is an integer, for all $t \leqslant 1$ both x and y are positive and for all $t \geqslant 2$ both x and y are negative. So xy will always be positive for all $t \in \mathbb{Z}$. Hence there is no t such that $xy \leqslant 0$.

Challenge

1 If one or both of a and b are zero, $\gcd(a,b) = \gcd(a+bc,b)$ is true.
Suppose that a and b are non-zero.
Then $\gcd(a,b) \mid a, b, a+bc$ and therefore $\gcd(a,b) \leqslant \gcd(a+bc,b)$
Similarly, $\gcd(a+bc,b) \mid a+bc, b$ and $a = (a+bc) - c \times b$, so $\gcd(a+bc,b) \leqslant \gcd(a,b)$
Hence $\gcd(a,b) = \gcd(a+bc,b)$

2 $a, b, d, x, y, z \in \mathbb{Z}$, and $\gcd(a,b) = d$, so $d > 0$
Given $z = ax + by$, $d \mid a, d \mid b \Rightarrow d \mid z$, and therefore $z \geqslant d$