Taylor Series 6A

1 a $f(x) = \sqrt{x} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f''''(a)}{4!}(x-a)^4 + \ldots$, where $a = 1$

$f(x) = \sqrt{x}$ $\quad f(1) = 1$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ $\quad f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$ $\quad f'''(1) = \frac{3}{8}$

$f''''(x) = -\frac{15}{16}x^{-\frac{7}{2}}$ $\quad f''''(1) = -\frac{15}{16}$

$$\text{So } \sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{4\times 2!}(x-1)^2 + \frac{3}{8\times 3!}(x-1)^3 - \frac{15}{16\times 4!}(x-1)^4 + \ldots$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \ldots$$

b $$\sqrt{1.2} \approx 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 + \frac{1}{16}(0.2)^3 - \frac{5}{128}(0.2)^4$$

$$\approx 1 + 0.1 - 0.005 + 0.0005 - 0.0000625$$

$$= 1.095 \text{ (3 d.p.)}$$

2 All solutions use the Taylor expansion in the form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \frac{f''''(a)(x-a)^4}{4!} + \ldots$$

a Let $f(x) = \ln x$ then $f(a) = f(e) = \ln e = 1$

$f'(x) = \frac{1}{x}$ $\quad f'(a) = f'(e) = \frac{1}{e}$

$f''(x) = -\frac{1}{x^2}$ $\quad f''(a) = f''(e) = -\frac{1}{e^2}$

$$\text{So } f(x) = \ln x = 1 + \frac{1}{e}(x-e) + \frac{\left(-\frac{1}{e^2}\right)}{2!}(x-e)^2 + \ldots$$

$$= 1 + \frac{(x-e)}{e} - \frac{(x-e)^2}{2e^2} + \ldots$$

2 b Let $f(x) = \tan x$ then $f(a) = f\left(\frac{\pi}{3}\right) = \sqrt{3}$

$f'(x) = \sec^2 x$ $\quad f'(a) = f'\left(\frac{\pi}{3}\right) = 4$

$f'(x) = 2\sec^2 x \tan x$ $\quad f''(a) = f''\left(\frac{\pi}{3}\right) = 2(4)(\sqrt{3}) = 8\sqrt{3}$

$f'''(x) = 2\sec^4 x + 2\tan x(2\sec^2 x \tan x)$ $\quad f'''(\text{a}) = f'''\left(\frac{\pi}{3}\right) = 2(16) + 4(4)(3) = 80$

$$\text{So } f(x) = \tan x = \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + \frac{8\sqrt{3}}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{80}{3!}\left(x - \frac{\pi}{3}\right)^3 + \ldots$$

$$= \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + 4\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{3}\right)^3 + \ldots$$

c Let $f(x) = \cos x$ then $f(a) = f(1) = \cos 1$

$f'(x) = -\sin x$ $\quad f'(a) = f'(1) = -\sin 1$

$f''(x) = -\cos x$ $\quad f''(a) = f''(1) = -\cos 1$

$f'''(x) = \sin x$ $\quad f'''(a) = f'''(1) = \sin 1$

$f''''(x) = \cos x$ $\quad f''''(a) = f''''(1) = \cos 1$

$$\text{So } f(x) = \cos x = \cos 1 - \sin 1(x-1) - \frac{(\cos 1)}{2}(x-1)^2 + \frac{(\sin 1)}{6}(x-1)^3 + \frac{(\cos 1)}{24}(x-1)^4 + \ldots$$

3 a i Let $f(x) = \cos(x)$. Then $\cos\left(x + \frac{\pi}{4}\right) = f\left(x + \frac{\pi}{4}\right)$

$f'(x) = \cos(x) \Rightarrow f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin(x) \Rightarrow f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f''(x) = -\cos(x) \Rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = \sin(x) \Rightarrow f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''''(x) = \cos(x) \Rightarrow f''''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\text{Using } f(x+a) = f(a) + f'(a)\,x + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!} + \frac{f''''(a)x^4}{4!} + \ldots$$

$$\cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 - x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \ldots\right)$$

3 a ii Let $\mathrm{f}(x) = \ln(x)$. Then $\ln(x+5) = \mathrm{f}(x+5)$

$\mathrm{f}(x) = \ln(x) \Rightarrow \mathrm{f}(5) = \ln 5$

$\mathrm{f}'(x) = \dfrac{1}{x} \Rightarrow \mathrm{f}'(5) = \dfrac{1}{5}$

$\mathrm{f}''(x) = -\dfrac{1}{x^2} \Rightarrow \mathrm{f}''(5) = -\dfrac{1}{25}$

$\mathrm{f}'''(x) = \dfrac{2}{x^3} \Rightarrow \mathrm{f}'''(5) = \dfrac{2}{125}$

$\mathrm{f}''''(x) = -\dfrac{6}{x^4} \Rightarrow \mathrm{f}''''(5) = -\dfrac{6}{625}$

Using $\mathrm{f}(x+a) = \mathrm{f}(a) + \mathrm{f}'(a)x + \dfrac{\mathrm{f}''(a)x^2}{2!} + \dfrac{\mathrm{f}'''(a)x^3}{3!} + \dfrac{\mathrm{f}''''(a)x^4}{4!} + \ldots$

$\ln(x+5) = \ln 5 + \dfrac{1}{5}x - \dfrac{1}{50}x^2 + \dfrac{1}{375}x^3 - \dfrac{1}{2500}x^4 + \ldots$

iii Let $\mathrm{f}(x) = \sin(x)$. Then $\sin\left(x - \dfrac{\pi}{3}\right) = \mathrm{f}\left(x - \dfrac{\pi}{3}\right)$

$\mathrm{f}(x) = \sin(x) \Rightarrow \mathrm{f}\left(-\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

$\mathrm{f}'(x) = \cos(x) \Rightarrow \mathrm{f}'\left(-\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

$\mathrm{f}''(x) = -\sin(x) \Rightarrow \mathrm{f}''\left(-\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

$\mathrm{f}'''(x) = -\cos(x) \Rightarrow \mathrm{f}'''\left(-\dfrac{\pi}{3}\right) = -\dfrac{1}{2}$

$\mathrm{f}''''(x) = \sin(x) \Rightarrow \mathrm{f}''''\left(-\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

Using $\mathrm{f}(x+a) = \mathrm{f}(a) + \mathrm{f}'(a)x + \dfrac{\mathrm{f}''(a)x^2}{2!} + \dfrac{\mathrm{f}'''(a)x^3}{3!} + \dfrac{\mathrm{f}''''(a)x^4}{4!} + \ldots$

$$\sin\left(x - \frac{\pi}{3}\right) = \frac{1}{2}\left(-\sqrt{3} + x + \frac{\sqrt{3}}{2!}x^2 - \frac{1}{3!}x^3 - \frac{\sqrt{3}}{4!}x^4 + \ldots\right)$$
$$= \frac{1}{2}\left(-\sqrt{3} + x + \frac{\sqrt{3}}{2}x^2 - \frac{1}{6}x^3 - \frac{\sqrt{3}}{24}x^4 + \ldots\right)$$

b Taking $x = 0.2$,

$\ln 5.2 \approx \ln 5 + \dfrac{0.2}{5} - \dfrac{0.2^2}{50} + \dfrac{0.2^3}{375} - \dfrac{0.2^4}{2500}$

≈ 1.649 to 4 s.f.

4 a $y = xe^x, \dfrac{dy}{dx} = xe^x + e^x = e^x(x+1)$ Product rule.

$$\frac{d^2y}{dx^2} = xe^x + e^x + e^x = e^x(x+2)$$

$$\frac{d^3y}{dx^3} = xe^x + 2e^x + e^x = e^x(x+3)$$

Each differentiation adds another e^x, so $\dfrac{d^ny}{dx^n} = (n+x)e^x$

So for $f(x) = xe^x, f^{(n)}(x) = (n+x)e^x$

b Using the Taylor series with $a = -1$, $f(-1) = -e^{-1}$, $f'(-1) = 0, f''(-1) = e^{-1}$

$f'''(-1) = 2e^{-1}$, $f''''(-1) = 3e^{-1}$

$$\text{So } xe^x = e^{-1}\left\{-1 + 0(x+1) + \frac{1}{2!}(x+1)^2 + \frac{2}{3!}(x+1)^3 + \frac{3}{4!}(x+1)^4 + \ldots\right\}$$

$$= e^{-1}\left\{-1 + \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 + \frac{1}{8}(x+1)^4 + \ldots\right\}$$

5 a Let $f(x) = x^3 \ln x$ then as $a = 1$ $f(a) = f(1) = 0$

$f'(x) = 3x^2 \ln x + x^3 \times \dfrac{1}{x} = x^2(1 + 3\ln x)$ $f'(a) = f'(1) = 1$

$f''(x) = x^2 \times \dfrac{3}{x} + 2x(1 + 3\ln x) = x(5 + 6\ln x)$ $f''(a) = f''(1) = 5$

$f'''(x) = x \times \dfrac{6}{x} + (5 + 6\ln x) = (11 + 6\ln x)$ $f'''(a) = f'''(1) = 11$

$f''''(x) = \dfrac{6}{x}$ $f''''(a) = f''''(1) = 6$

Using Taylor, form **ii**

$$f(x) = x^3 \ln x = 0 + 1(x-1) + \frac{5}{2!}(x-1)^2 + \frac{11}{3!}(x-1)^3 + \frac{6}{4!}(x-1)^4 + \ldots$$

$$= (x-1) + \frac{5}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \frac{1}{4}(x-1)^4 + \ldots$$

b Substituting $x = 1.5$ in series in **a**, gives

$$\frac{27}{8}\ln 1.5 \approx 0.5 + \frac{5}{2}(0.5)^2 + \frac{11}{6}(0.5)^s + \frac{1}{4}(0.5)^4 + \ldots$$

$$\approx 0.5 + 0.625 + 0.22916\ldots + 0.015625 (= 1.369791\ldots)$$

So this gives an approximation for $\ln 1.5$ of $\dfrac{8}{27}(1.369791\ldots) = 0.4059$ (4 d.p.)

6 We look to find the Taylor series for the -function $\tan(x-\alpha)$ about $x=0$ where, $\alpha = \arctan\left(\frac{3}{4}\right)$

Let $f(x)=\tan(x-\alpha)$ then;

$f(0)=\tan(-\alpha)=-\tan(\alpha)=-\frac{3}{4}$

$f'(x)=\sec^2(x-\alpha) \Rightarrow f'(0)=\sec^2(-\alpha)=\sec^2\alpha=\frac{25}{16}$

Now we use the fact that;

$f''(x)=2\tan(x-\alpha)\sec^2(x-\alpha) \Rightarrow f''(0)=-2\tan\alpha\sec^2\alpha$

$\Rightarrow f''(0)=-2\cdot\frac{3}{4}\cdot\frac{25}{16}=-\frac{75}{32}$

Finally, using $f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\ldots(x^3)$, we find that

$f(x)=\tan(x-\alpha)=-\frac{3}{4}+\frac{25}{16}\cdot x+\frac{1}{2!}\cdot\frac{-75}{32}\cdot x^2+\ldots(x^3)$

$\Rightarrow \tan(x-\alpha)=-\frac{3}{4}+\frac{25}{16}x-\frac{75}{64}x^2+\ldots(x^3)$

7 Let $f(x)=\sin 2x$, then we calculate the required derivatives;

$f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$f'(x)=2\cos 2x \Rightarrow f'\left(\frac{\pi}{6}\right)=2\cos\frac{\pi}{3}=1$

$f''(x)=-4\sin 2x \Rightarrow f''\left(\frac{\pi}{6}\right)=-4\sin\frac{\pi}{3}=-\frac{4\sqrt{3}}{2}=-2\sqrt{3}$

$f'''(x)=-8\cos 2x \Rightarrow f'''\left(\frac{\pi}{6}\right)=-8\cos\frac{\pi}{3}=-4$

$f^{(4)}(x)=16\sin 2x \Rightarrow f^{(4)}\left(\frac{\pi}{6}\right)=8\sqrt{3}$

Then, using the Taylor expansion:
$$f(x)=f\left(\frac{\pi}{6}\right)+f'\left(\frac{\pi}{6}\right)(x-\tfrac{\pi}{6})+\frac{1}{2!}f''\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right)^2$$
$$+\tfrac{1}{3!}f'''\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right)^3+\tfrac{1}{4!}f^{(4)}\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right)^4+\ldots\left(\left(x-\frac{\pi}{6}\right)^5\right)$$

We find that

$$\sin 2x=\frac{\sqrt{3}}{2}+1\cdot\left(x-\frac{\pi}{6}\right)+\frac{-2\sqrt{3}}{2!}\cdot(x-\tfrac{\pi}{6})^2$$
$$+\frac{-4}{3!}\cdot\left(x-\frac{\pi}{6}\right)^3+\frac{8\sqrt{3}}{4!}\cdot\left(x-\frac{\pi}{6}\right)^4+\ldots\left(\left(x-\frac{\pi}{6}\right)^5\right)$$
$$\Rightarrow \sin 2x=\frac{\sqrt{3}}{2}+\left(x-\frac{\pi}{6}\right)-\sqrt{3}\left(x-\frac{\pi}{6}\right)^2$$
$$-\frac{2}{3}\left(x-\frac{\pi}{6}\right)^3+\frac{\sqrt{3}}{3}\left(x-\frac{\pi}{6}\right)^4+\ldots\left(\left(x-\frac{\pi}{6}\right)^5\right)$$

8 **a** Given $y = \dfrac{1}{\sqrt{(1+x)}} = (1+x)^{-\frac{1}{2}}$ $\qquad y_3(=\text{value of } y \text{ when } x = 3) = \dfrac{1}{2}$

$$\frac{dy}{dx} = -\frac{1}{2}(1+x)^{-\frac{3}{2}} \qquad \left(\frac{dy}{dx}\right)_3 = -\frac{1}{2}\times\frac{1}{8} = -\frac{1}{16}$$

$$\frac{d^2y}{dx^2} = \frac{3}{4}(1+x)^{-\frac{5}{2}} \qquad \left(\frac{d^2y}{dx^2}\right)_3 = \frac{3}{4}\times\frac{1}{32} = \frac{3}{128}$$

b So using

$$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \ldots \qquad \text{with } f^{(n)}(3) \equiv \left(\frac{d^n y}{dx^n}\right)_3$$

$$y = \frac{1}{\sqrt{(1+x)}} = \frac{1}{2} - \frac{1}{16}(x-3) + \frac{3}{256}(x-3)^2 + \ldots$$

9 Let $f(x) = \cosh x$, then note the following

$$\cosh(\ln 5) = \frac{1}{2}(e^{\ln 5} + e^{-\ln 5}) = \frac{1}{2}\left(5 + \frac{1}{5}\right) = \frac{13}{5}$$

$$\sinh(\ln 5) = \frac{1}{2}(e^{\ln 5} - e^{-\ln 5}) = \frac{1}{2}\left(5 + \frac{1}{5}\right) = \frac{12}{5}$$

Now we calculate the required derivatives:

$$f(x) = \cosh x \Rightarrow f(\ln 5) = \frac{13}{5}$$

$$f'(x) = \sinh x \Rightarrow f'(\ln 5) = \frac{12}{5}$$

$$f''(x) = \cosh x \Rightarrow f''(\ln 5) = \frac{13}{5}$$

$$f'''(x) = \sinh x \Rightarrow f'''(\ln 5) = \frac{12}{5}$$

$$f^{(4)}(x) = \cosh x \Rightarrow f^{(4)}(\ln 5) = \frac{13}{5}$$

Then we can use:

$$f(x) = f(\ln 5) + f'(\ln 5)(x - \ln 5) + \frac{1}{2!}f''(\ln 5)(x-\ln 5)^2$$
$$+ \frac{1}{3!}f'''(\ln 5)(x-\ln 5)^3 + \frac{1}{4!}f^{(4)}(\ln 5)(x-\ln 5)^4$$
$$+ \ldots$$

To find:

$$\cosh x = \frac{13}{5} + \frac{12}{5}(x-\ln 5) + \frac{1}{2!}\cdot\frac{13}{5}(x-\ln 5)^2$$
$$+ \frac{1}{3!}\cdot\frac{12}{5}(x-\ln 5)^3 + \frac{1}{4!}\cdot\frac{13}{5}(x-\ln 5)^4 + \ldots$$

$$\Rightarrow \cosh x = \frac{13}{5} + \frac{12}{5}(x-\ln 5) + \frac{13}{10}(x-\ln 5)^2$$
$$+ \frac{2}{5}(x-\ln 5)^3 + \frac{13}{120}(x-\ln 5)^4 + \ldots$$

10 a Consider the first two terms in the Taylor series of a function $\mathrm{f}(x)$ about $x = \ln 2$:

$$\mathrm{f}(x) = \mathrm{f}(\ln 2) + \mathrm{f}'(\ln 2)(x - \ln 2) + \ldots$$

So we see that with $\mathrm{f}(x) = \sinh ax$, it must be the case that $\mathrm{f}'(\ln 2) = \frac{17}{4}$, so:

$$\mathrm{f}'(\ln 2) = a\cosh(a\ln 2) = \frac{1}{2}a(e^{a\ln 2} + e^{-a\ln 2})$$

$$\Rightarrow \frac{17}{4} = \frac{1}{2}a(e^{a\ln 2} + e^{-a\ln 2}) = \frac{1}{2}a(e^{\ln 2^a} + e^{\ln 2^{-a}})$$

$$\Rightarrow \frac{17}{4} = \frac{1}{2}a\left(2^a + \frac{1}{2^a}\right) \Rightarrow 17 = 2a\left(2^a + \frac{1}{2^a}\right)$$

$$2 \cdot 2 \cdot (2^2 + 2^{-2}) = 17 \Rightarrow a = 2$$

b With $a = 2$ we note that:

$$\sinh(2\ln 2) = \frac{1}{2}(e^{2\ln 2} - e^{-2\ln 2}) = \frac{1}{2}(e^{\ln 4} - e^{-\ln 4})$$

$$\Rightarrow \sinh(2\ln 2) = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$$

$$\cosh(2\ln 2) = \frac{1}{2}(e^{2\ln 2} + e^{-2\ln 2}) = \frac{1}{2}(e^{\ln 4} + e^{-\ln 4})$$

$$\Rightarrow \cosh(2\ln 2) = \frac{1}{2}\left(4 + \frac{1}{4}\right) = \frac{17}{8}$$

Then we can calculate the required derivatives:

$$\mathrm{f}(\ln 2) = \sinh(2\ln 2) = \frac{15}{8}$$

$$\mathrm{f}'(x) = 2\cosh 2x \Rightarrow \mathrm{f}'(\ln 2) = \frac{17}{4}$$

$$\mathrm{f}''(x) = 4\sinh 2x \Rightarrow \mathrm{f}''(\ln 2) = \frac{15}{2}$$

$$\mathrm{f}'''(x) = 8\cosh 2x \Rightarrow \mathrm{f}'''(\ln 2) = 17$$

Then we can use:

$$\mathrm{f}(x) = \mathrm{f}(\ln 2) + \mathrm{f}'(\ln 2)(x - \ln 2) + \frac{1}{2!}\mathrm{f}''(\ln 2)(x - \ln 2)^2$$
$$+ \frac{1}{3!}\mathrm{f}'''(\ln 2)(x - \ln 2)^3 + \ldots$$

To find that:

$$\sinh 2x = \frac{15}{8} + \frac{17}{4}(x - \ln 2) + \frac{1}{2!} \cdot \frac{15}{2}(x - \ln 2)^2$$
$$+ \frac{1}{3!} \cdot 17(x - \ln 2)^3 + \ldots\left((x - \ln 2)^4\right)$$

$$\Rightarrow \sinh 2x = \frac{15}{8} + \frac{17}{4}(x - \ln 2) + \frac{15}{4}(x - \ln 2)^2$$
$$+ \frac{17}{6}(x - \ln 2)^3 + \ldots$$

11 Consider the Taylor expansion for a function $f(x)$ about the point $x = 2$:

$$f(x) = f(2) + \sum_{k=1}^{\infty} \frac{1}{k!} f^{(k)}(2) \cdot (x-2)^k$$

Here, $f(x) = \ln x$, so $f(2) = \ln 2$

Then, the subsequent derivatives are:

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f'''(x) = \frac{2}{x^3},$$

$$f^{(4)}(x) = -\frac{3 \cdot 2}{x^4}, \ldots, f^{(k)}(x) = (-1)^{k-1} \frac{k!}{k \cdot x^k}$$

$$\Rightarrow f^{(k)}(2) = (-1)^{k-1} \frac{k!}{k \cdot 2^k}$$

Plugging this into the Taylor series, we see that:

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{1}{n!} \cdot (-1)^{n-1} \frac{n!}{n \cdot 2^n} (x-2)^n$$

$$\Rightarrow \ln x = \ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} (x-2)^n$$

Challenge

a Let $f(x) = \ln(\cos 2x)$, then we will use:

$$f(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{1}{2!} f''(\pi)(x-\pi)^2$$
$$+ \frac{1}{3!} f'''(\pi)(x-\pi)^3 + \ldots \left((x-\pi)^4\right)$$

Differentiating:

$$f(\pi) = \ln(\cos 2\pi) = 0$$

$$f'(x) = -\frac{2\sin 2x}{\cos 2x} = -2\tan 2x \Rightarrow f'(\pi) = 0$$

$$f''(x) = -4\sec^2 2x \Rightarrow f''(\pi) = -\frac{4}{1^2} = -4$$

$$f'''(x) = -16\tan 2x \sec^2 2x \Rightarrow f'''(\pi) = 0$$

So, $\ln(\cos 2x) = -2(x-\pi)^2 - \frac{4}{3}(x-\pi)^4 - \ldots$

b Note that $\cos\left(2 \cdot \frac{13\pi}{12}\right) = \frac{\sqrt{3}}{2}$, so we take the leading order term in the expansion above setting $x = \frac{13\pi}{12}$

$$\ln\left(\cos \frac{13\pi}{6}\right) = \ln \frac{\sqrt{3}}{2} \approx -2\left(\frac{13\pi}{12} - \pi\right)^2 = -0.1433 \ (4\,\text{d.p.})$$