

Inequalities Mixed Exercise 4

- 1 To ensure multiplication by a positive quantity, multiply both sides by $(x-2)^2 x^2$

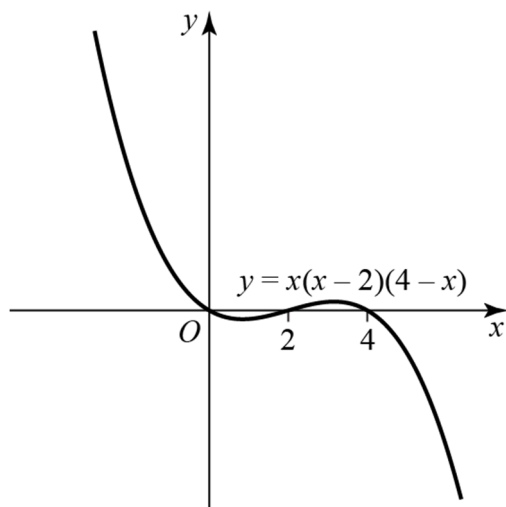
$$\frac{1}{\cancel{(x-2)}} \times (x-2)^{\cancel{2}} x^2 \leq \frac{2}{\cancel{x}} \times (x-2)^2 x^{\cancel{2}}$$

$$x(x-2)(x-2(x-2)) \leq 0$$

$$x(x-2)(4-x) \leq 0$$

So the critical values are $x = 0, 2$ or 4

The curve $y = x(x-2)(4-x)$ is a cubic graph with negative x^3 coefficient, so the curve starts in the top left and ends in the bottom right and passes through $(0,0)$, $(2,0)$ and $(4,0)$.



The solution corresponds to the section of the graph that is on or below the x -axis.
So the solution is $0 \leq x \leq 2$ or $x \geq 4$

2 To ensure multiplication by a positive quantity, multiply both sides by $(x+2)^2$

$$\frac{2x^2-2}{(x+2)} \times (x+2)^2 > 4(x+2)^2$$

$$(x+2)((2x^2-2)-4(x+2)) > 0$$

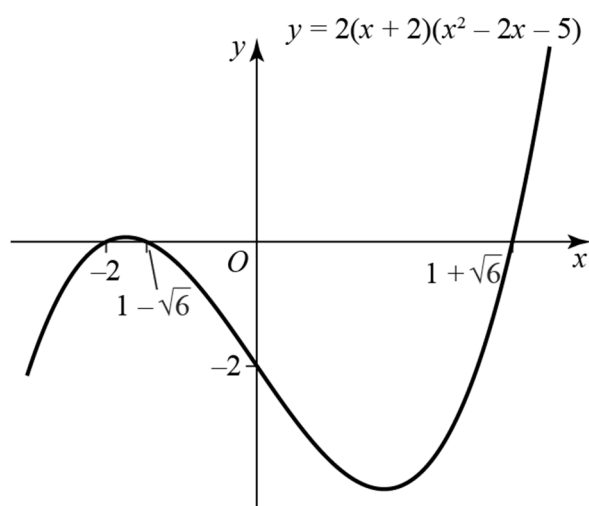
$$(x+2)(2x^2-4x-10) > 0$$

$$2(x+2)(x^2-2x-5) > 0$$

So using the quadratic formula the critical values are $x = -2$ or $\frac{2 \pm \sqrt{24}}{2}$

This simplifies to $x = -2$ or $1 \pm \sqrt{6}$

The curve $y = 2(x+2)(x^2-2x-5)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(-2,0)$, $(1-\sqrt{6},0)$ and $(1+\sqrt{6},0)$.



The solution corresponds to the section of the graph that is above the x -axis.

So the solution is $-2 < x < 1 - \sqrt{6}$ or $x > 1 + \sqrt{6}$

- 3 To ensure multiplication by a positive quantity, multiply both sides by $(x-2)^2$

$$\frac{2x^2 - 3x + 4}{\cancel{(x-2)}} \times (x-2)^{\cancel{2}} < (4x-2)(x-2)^2$$

$$(x-2)((2x^2 - 3x + 4) - (4x-2)(x-2)) < 0$$

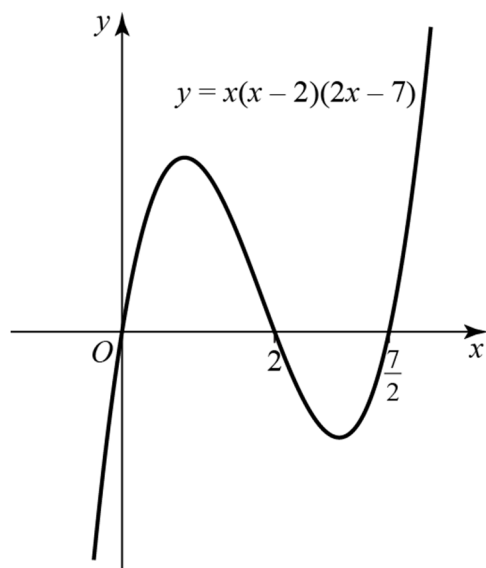
$$(x-2)(-2x^2 + 7x) < 0$$

$$x(x-2)(7-2x) < 0$$

$$x(x-2)(2x-7) > 0 \quad \text{multiplying by } -1 \text{ so changing the direction of the inequality}$$

So the critical values are $x = 0, 2$ or $\frac{7}{2}$

The curve $y = x(x-2)(2x-7)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(0,0)$, $(2,0)$ and $(\frac{7}{2}, 0)$.



The solution corresponds to the section of the graph that is above the x -axis.

So the solution is $0 < x < 2$ or $x > \frac{7}{2}$

4 To ensure multiplication by a positive quantity, multiply both sides by $(2x-3)^2(x-3)^2$

$$\frac{x+1}{\cancel{(2x-3)}} \times (2x-3)^{\cancel{2}}(x-3)^{\cancel{2}} < \frac{1}{\cancel{(x-3)}} \times (2x-3)^{\cancel{2}}(x-3)^{\cancel{2}}$$

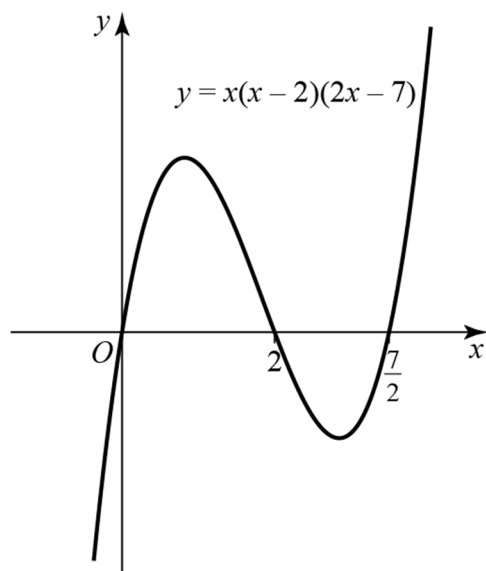
$$(2x-3)(x-3)((x+1)(x-3)-(2x-3)) < 0$$

$$(2x-3)(x-3)(x^2-4x) < 0$$

$$x(2x-3)(x-3)(x-4) < 0$$

So the critical values are $x = 0, \frac{3}{2}, 3$ or 4

The curve $y = x(2x-3)(x-3)(x-4)$ is a quartic graph with positive x^4 coefficient, so the curve starts in the top left and ends in the top right and passes through $(0,0), (\frac{3}{2},0), (3,0)$ and $(4,0)$.



The solution corresponds to the section of the graph that is below the x -axis.

So the solution in set notation is $\left\{x: 0 < x < \frac{3}{2}\right\} \cup \{x: 3 < x < 4\}$

5 To ensure multiplication by a positive quantity, multiply both sides by $(x-1)^2$

$$\frac{(x+3)(x+9)}{\cancel{(x-1)}} \times (x-1)^2 > (3x-5)(x-1)^2$$

$$(x-1)((x+3)(x+9) - (3x-5)(x-1)) > 0$$

$$(x-1)(x^2 + 12x + 27) - (3x^2 - 8x + 5) > 0$$

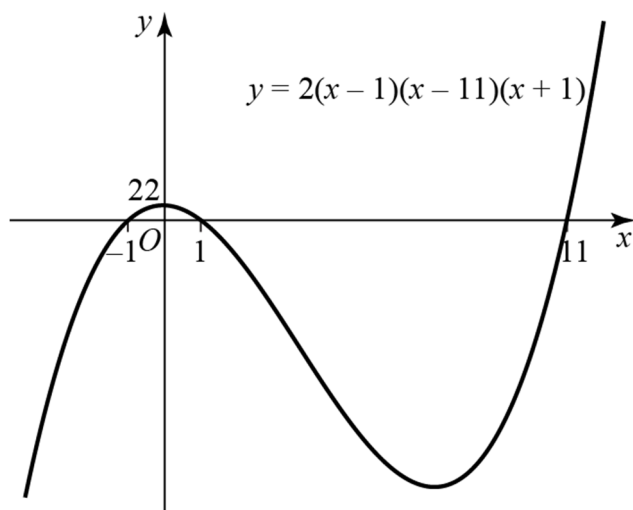
$$(x-1)(-2x^2 + 20x + 22) > 0$$

$$2(x-1)(x^2 - 10x - 11) < 0 \quad \text{multiplying by } -1 \text{ so changing the direction of the inequality}$$

$$2(x-1)(x-11)(x+1) < 0$$

So the critical values are $x = -1, 1$ or 11

The curve $y = 2(x-1)(x-11)(x+1)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(-1,0)$, $(1,0)$ and $(11,0)$.



The solution corresponds to the section of the graph that is below the x -axis.

So the solution in set notation is $\{x: x < -1\} \cup \{x: 1 < x < 11\}$

6 a $y = 2x + 2$

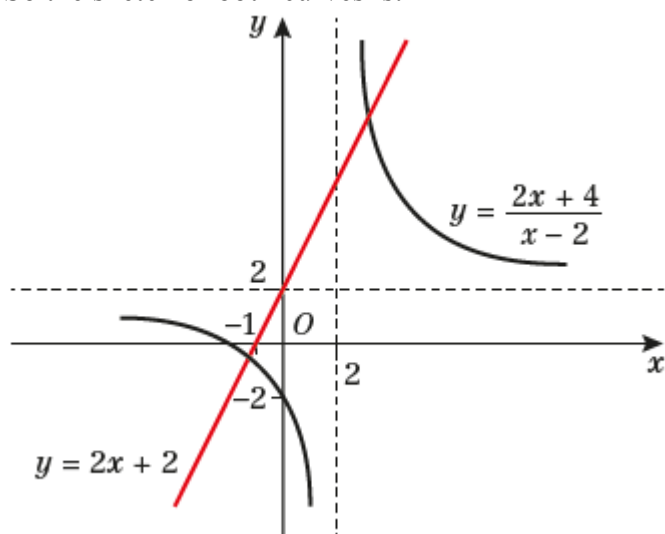
The graph is a straight line with a positive gradient that passes through $(-1, 0)$ and $(0, 2)$

$$y = \frac{2x+4}{x-2}$$

$$y = 2\left(\frac{x+2}{x-2}\right) = 2\left(\frac{x-2+4}{x-2}\right) = 2\left(1 + \frac{4}{x-2}\right) \quad \text{rearranging to see how the curve behaves as } x \rightarrow \infty$$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 2$ (as $x \rightarrow \pm\infty$, $y \rightarrow 2$) and a vertical asymptote at $x = 2$ (as $x \rightarrow 2$, $y \rightarrow \pm\infty$). The graph crosses the y -axis at $(0, -2)$.

So the sketch of both curves is:



- b** The x -coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$2x + 2 = \frac{2x + 4}{x - 2}$$

$$(x + 1)(x - 2) = x + 2$$

$$x^2 - x - 2 = x + 2$$

$$x^2 - 2x - 4 = 0$$

$$\Rightarrow x = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5} \quad \text{using the quadratic formula}$$

The solution to the inequality is when the line $y = 2x + 2$ lies above the curve $y = \frac{2x + 4}{x - 2}$

Using the sketch from part **a** and the x -coordinates of the points of intersection this occurs when

$$1 - \sqrt{5} < x < 2 \quad \text{or} \quad x > 1 + \sqrt{5}$$

7 a $y = 2 - 4x$

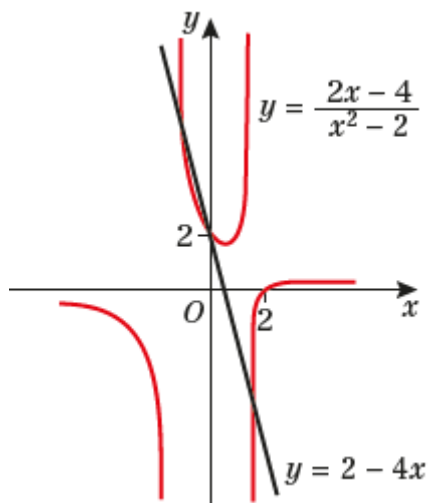
The graph is a straight line with a negative gradient that passes through $(0, 2)$ and $(\frac{1}{2}, 0)$

$$y = \frac{2x - 4}{x^2 - 2}$$

The graph crosses the y -axis at $(0, 2)$ and the x -axis at $(2, 0)$. There are vertical asymptotes at $x = \sqrt{2}$ and $x = -\sqrt{2}$. There is a horizontal asymptote at $y = 0$ (as $x \rightarrow \pm\infty$, $y \rightarrow 0$).

Note the regions where y is negative: $2x - 4 < 0$ for $x < 2$ and $x^2 - 2 < 0$ for $-\sqrt{2} < x < \sqrt{2}$ so $y < 0$ for $x < -\sqrt{2}$ and $\sqrt{2} < x < 2$

So the sketch of both curves is:



- b The x -coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$2 - 4x = \frac{2x - 4}{x^2 - 2}$$

$$(1 - 2x)(x^2 - 2) = x - 2$$

$$-2x^3 + x^2 + 4x - 2 = x - 2$$

$$2x^3 - x^2 - 3x = 0$$

$$x(2x^2 - x - 3) = 0$$

$$\Rightarrow x = 0, \frac{1 \pm \sqrt{1 + 24}}{4} \quad \text{using the quadratic formula}$$

$$\Rightarrow x = -1, 0 \text{ or } \frac{3}{2}$$

The solution to the inequality is when the line $y = 2 - 4x$ lies below the curve $y = \frac{2x - 4}{x^2 - 2}$

Using the sketch from part a and the x -coordinates of the points of intersection this occurs when

$$-\sqrt{2} < x < -1 \text{ or } 0 < x < \sqrt{2} \text{ or } x > \frac{3}{2}$$

8 a $y = \frac{x-2}{3x-1}$
 $y = \frac{x-2}{3x-1} = \frac{1}{3} \left(\frac{3x-1-5}{3x-1} \right) = \frac{1}{3} \left(1 - \frac{5}{3x-1} \right)$ rearranging to see how the curve behaves as $x \rightarrow \infty$

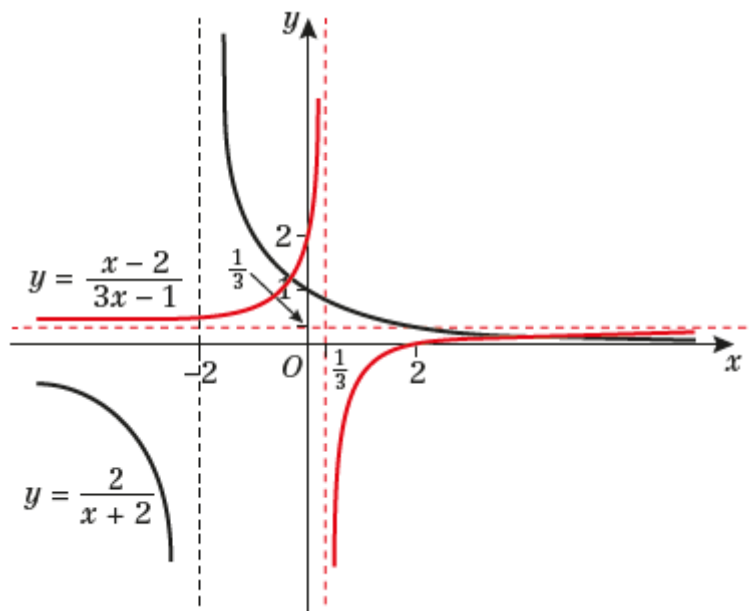
The curve is a reciprocal graph. There is a horizontal asymptote at $y = \frac{1}{3}$ (as $x \rightarrow \pm\infty$, $y \rightarrow \frac{1}{3}$) and a vertical asymptote at $x = \frac{1}{3}$ (as $x \rightarrow \frac{1}{3}$, $y \rightarrow \pm\infty$). The graph crosses the axes at $(0, 2)$ and $(2, 0)$.

$$y = \frac{2}{x+2}$$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 0$ (as $x \rightarrow \pm\infty$, $y \rightarrow 0$) and a vertical asymptote at $x = -2$ (as $x \rightarrow -2$, $y \rightarrow \pm\infty$). The graph crosses the axes at $(0, 1)$.

When $x < -2$, $y < 0$ and when $x > -2$, $y > 0$.

So the sketch of both curves is:



- b The x -coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$\frac{x-2}{3x-1} = \frac{2}{x+2}$$

$$(x-2)(x+2) = 2(3x-1)$$

$$x^2 - 4 = 6x - 2$$

$$x^2 - 6x - 2 = 0$$

$$\Rightarrow x = \frac{6 \pm \sqrt{36+8}}{2}$$

using the quadratic formula

$$\Rightarrow x = 3 \pm \sqrt{11}$$

The solution to the inequality is when the curve $y = \frac{x-2}{3x-1}$ lies below the curve $y = \frac{2}{x+2}$

Using the sketch from part a and the x -coordinates of the points of intersection this occurs when

$$-2 < x < 3 - \sqrt{11} \quad \text{or} \quad \frac{1}{3} < x < 3 + \sqrt{11}$$

$$9 \text{ a } y = \frac{x+1}{x-2}$$

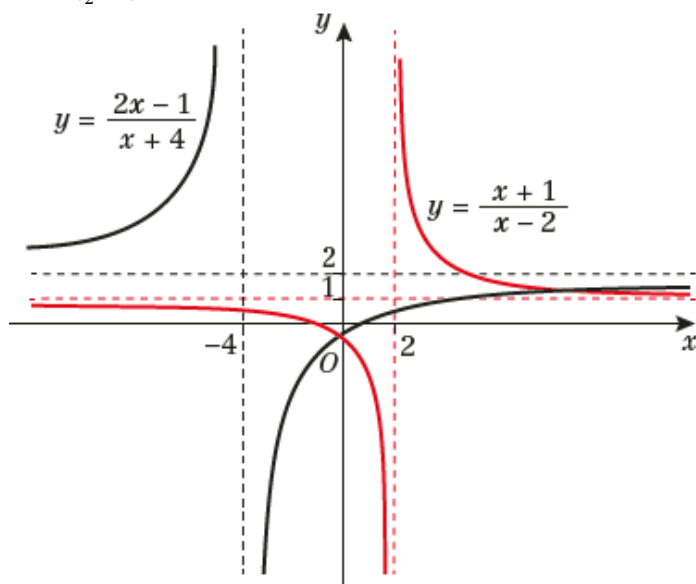
$$y = \frac{x+1}{x-2} = \frac{x-2+3}{x-2} = 1 + \frac{3}{x-2} \quad \text{rearranging to see how the curve behaves as } x \rightarrow \infty$$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 1$ (as $x \rightarrow \pm\infty$, $y \rightarrow 1$) and a vertical asymptote at $x = 2$ (as $x \rightarrow 2$, $y \rightarrow \pm\infty$). The graph crosses the axes at $(0, -\frac{1}{2})$ and $(-1, 0)$.

$$y = \frac{2x-1}{x+4}$$

$$y = \frac{2x-1}{x+4} = 2 \left(\frac{x+4-\frac{9}{2}}{x+4} \right) = 2 \left(1 - \frac{9}{2x+8} \right) \quad \text{rearranging to see how the curve behaves as } x \rightarrow \infty$$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 2$ (as $x \rightarrow \pm\infty$, $y \rightarrow 2$) and a vertical asymptote at $x = -4$ (as $x \rightarrow -4$, $y \rightarrow \pm\infty$). The graph crosses the axes at $(0, -\frac{1}{4})$ and $(\frac{1}{2}, 0)$. So the sketch of both curves is:



- b** The x -coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$\frac{x+1}{x-2} = \frac{2x-1}{x+4}$$

$$(x+1)(x+4) = (2x-1)(x-2)$$

$$x^2 + 5x + 4 = 2x^2 - 5x + 2$$

$$x^2 - 10x - 2 = 0$$

$$\Rightarrow x = \frac{10 \pm \sqrt{100+8}}{2}$$

using the quadratic formula

$$\Rightarrow x = 5 \pm \sqrt{27}$$

$$\Rightarrow x = 5 \pm 3\sqrt{3}$$

The solution to the inequality is when the curve $y = \frac{x+1}{x-2}$ lies below the curve $y = \frac{2x-1}{x+4}$

Using the sketch from part **a** and the x -coordinates of the points of intersection this occurs when

$$x < -4 \quad \text{or} \quad 5 - 3\sqrt{3} < x < 2 \quad \text{or} \quad x > 5 + 3\sqrt{3}$$

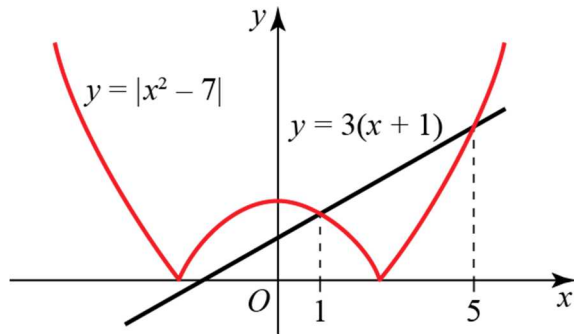
10 $y = |x^2 - 7|$

This is based on a parabola that is part reflected in the x -axis. It has a local maximum at $(0, 7)$ and it touches the the x -axis at $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$

$$y = 3(x + 1)$$

The graph is a straight line with a positive gradient that passes through $(-1, 0)$ and $(0, 3)$.

So the sketch of both curves is:



The critical values are given by

$$x^2 - 7 = 3x + 3$$

$$x^2 - 3x - 10 = 0$$

$$(x - 5)(x + 2) = 0$$

$$x = -2, 5$$

Or

$$-(x^2 - 7) = 3x + 3$$

$$x^2 + 3x - 4 = 0$$

$$(x + 4)(x - 1) = 0$$

$$x = -4, 1$$

From the sketch, the only valid critical values are $x = 1$ and $x = 5$.

The solution is when the line is above the curve

So the solution is $1 < x < 5$

11 Rearranging and simplifying gives

$$\frac{x^2}{|x|+6} < 1$$

$$x^2 < |x|+6 \quad \text{because } |x|+6 \text{ is always positive}$$

$$x^2 - 6 < |x|$$

The critical values are given by

$$x^2 - 6 = x$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = -2, 3$$

Or

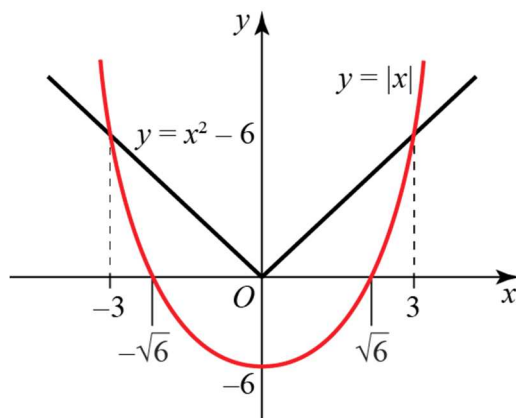
$$x^2 - 6 = -x$$

$$x^2 + x - 6 = 0$$

$$(x-2)(x+3) = 0$$

$$x = -3, 2$$

Sketching both curves gives:



From the sketch, the intersections are $> |\sqrt{6}|$, so the valid critical values are $x = -3$ and $x = 3$.

The solution is when the curve is below $y = |x|$

So the solution is $-3 < x < 3$

- 12 The critical values are given by

$$x - 1 = 6x - 1$$

$$x = 0$$

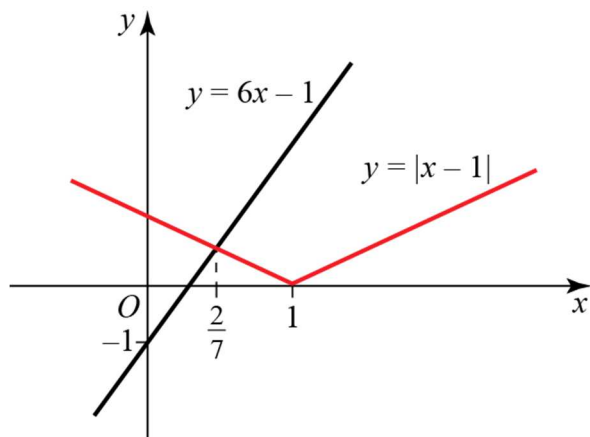
Or

$$-(x - 1) = 6x - 1$$

$$7x = 2$$

$$x = \frac{2}{7}$$

Sketching both curves gives:

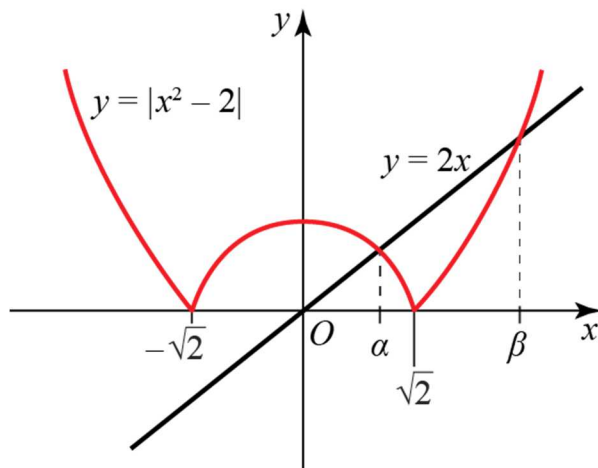


From the sketch, the only valid critical values is $x = \frac{2}{7}$

The solution is when the v-shaped curve is above the line

So the solution is $x < \frac{2}{7}$

- 13 Sketching $y = |x^2 - 2|$ and $y = 2x$ gives



The critical values are given by

$$\begin{aligned} x^2 - 2 &= 2x \\ x^2 - 2x - 2 &= 0 \\ x &= \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3} \quad \text{using the quadratic formula} \end{aligned}$$

Or

$$\begin{aligned} -(x^2 - 2) &= 2x \\ x^2 + 2x - 2 &= 0 \\ x &= \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3} \quad \text{using the quadratic formula} \end{aligned}$$

From the sketch, valid critical values are greater than 0, so the two valid critical values are $\alpha = \sqrt{3} - 1$ and $\beta = 1 + \sqrt{3}$

The solution is when the line is below the curve

So the solution is $x < \sqrt{3} - 1$ or $x > 1 + \sqrt{3}$

14 a $y = 5x - 1$

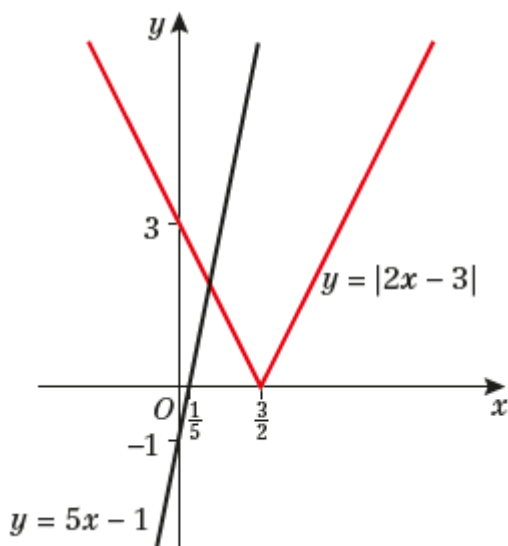
The graph is a straight line with a positive gradient that passes through $(0, -1)$ and $(\frac{1}{5}, 0)$

$$y = |2x - 3|$$

This function has a v-shaped graph that touches the x -axis at $(\frac{3}{2}, 0)$ and crosses that y -axis at $(0, 3)$.

The graph has a positive gradient for $x > \frac{3}{2}$ and a negative gradient for $x < \frac{3}{2}$

Sketching both curves gives:



b The critical values are given by

$$2x - 3 = 5x - 1$$

$$x = -\frac{2}{3}$$

From the sketch, this is not a valid critical value.

Or

$$-(2x - 3) = 5x - 1$$

$$x = \frac{4}{7}$$

The solution is when the v-shaped curve is below the line

So the solution is $x > \frac{4}{7}$

- 15 a** Consider, in turn, when the argument of the modulus function is positive and is negative.

$$2x^2 + x - 6 = 6 - 3x$$

$$2x^2 + 4x - 12 = 0$$

$$2(x^2 + 2x - 6) = 0$$

$$x = \frac{-2 \pm \sqrt{28}}{2}$$

using the quadratic formula

$$x = -1 \pm \sqrt{7}$$

$$-(2x^2 + x - 6) = 6 - 3x$$

$$2x^2 - 2x = 0$$

$$2x(x - 1) = 0$$

$$x = 0 \text{ or } 1$$

$$\text{Solution: } x = -1 - \sqrt{7}, 0, 1, -1 + \sqrt{7}$$

b $y = 2x^2 + x - 6 = (2x - 3)(x + 2)$

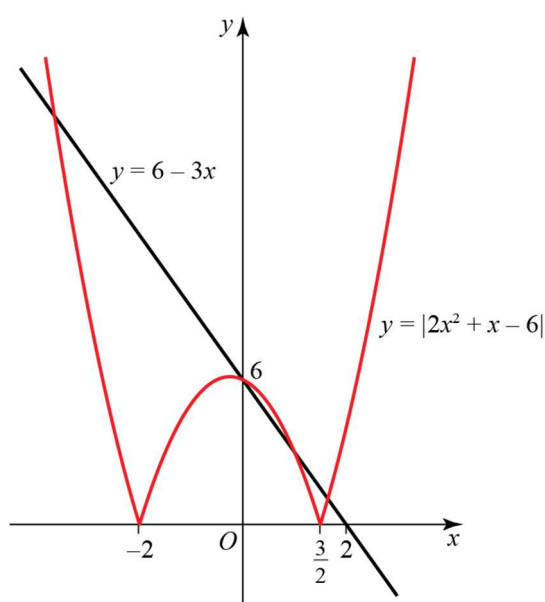
The curve is a quadratic graph with a positive x^2 coefficient, so it is a parabola with a minimum below the x -axis. The graph crosses the x -axis at $(-2, 0)$ and $(\frac{3}{2}, 0)$ and the y -axis at $(0, -6)$.

So the graph of $y = |2x^2 + x - 6|$ will be the graph of $y = 2x^2 + x - 6$ but with the section of the latter curve that is below the x -axis reflected in the x -axis. The graph crosses the x -axis at $(-2, 0)$ and $(\frac{3}{2}, 0)$ and the y -axis at $(0, 6)$.

$$y = 6 - 3x$$

The graph is a straight line with a negative gradient that passes through $(0, 6)$ and $(2, 0)$.

So a sketch of both curves is:



- c** The solution is when the curve is above the line

$$\text{So the solution is } x < -1 - \sqrt{7} \text{ or } 0 < x < 1 \text{ or } x > \sqrt{7} - 1$$

16 a $y = |x^2 - 4|$

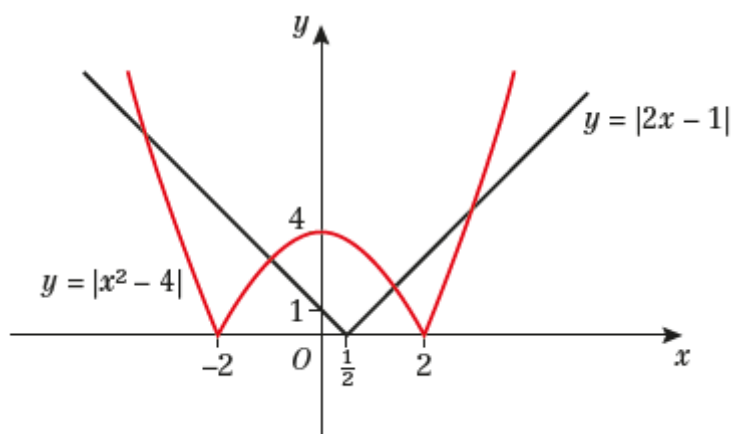
The graph of $y = |x^2 - 4|$ will be the graph of $y = x^2 - 4$ but with the section of the latter curve that is below the x -axis reflected in the x -axis. The graph crosses the x -axis at $(-2, 0)$ and $(2, 0)$ and the y -axis at $(0, 4)$, where there is a local maximum

$$y = |2x - 1|$$

This function has a v-shaped graph that touches the x -axis at $(\frac{1}{2}, 0)$ and crosses that y -axis at $(0, 1)$.

The graph has a positive gradient for $x > \frac{1}{2}$ and a negative gradient for $x < \frac{1}{2}$

So a sketch of both curves is:



- b** Consider, in turn, when the argument of the modulus function is positive and is negative.

$$x^2 - 4 = 2x - 1$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = -1, 3$$

$$-(x^2 - 4) = 2x - 1$$

$$x^2 + 2x - 5 = 0$$

$$x = \frac{-2 \pm \sqrt{24}}{2}$$

using the quadratic formula

$$x = -1 \pm \sqrt{6}$$

Solution: $x = -1 - \sqrt{6}, -1, -1 + \sqrt{6}, 3$

- c** The solution is when the v-shaped graph is below the curve

So the solution is $x < -1 - \sqrt{6}$ or $-1 < x < -1 + \sqrt{6}$ or $x > 3$

17 a The student has not sketched the curves to check whether the critical values identified are points of intersection. Some of the values are not valid critical values, and therefore the student has not found the correct solution for the inequality.

b $y = 3x + 2$

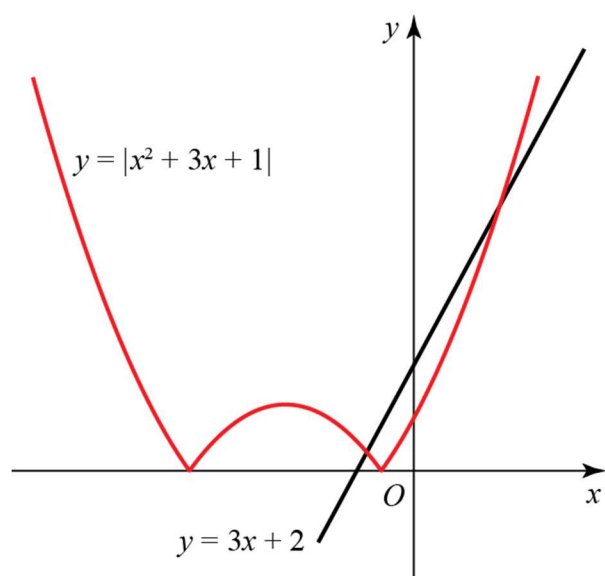
The graph is a straight line with a positive gradient that passes through $(0, 2)$ and $(-\frac{2}{3}, 0)$

$$y = x^2 + 3x + 1$$

The curve is a quadratic graph with a positive x^2 coefficient. It cuts the y -axis at $(0, 1)$. It cuts the x -axis when $x = \frac{1}{2}(-3 \pm \sqrt{5})$.

So the graph of $y = |x^2 + 3x + 1|$ will be the graph of $y = x^2 + 3x + 1$ but with the section of the latter curve that is below the x -axis reflected in the x -axis.

So a sketch of both curves is:



So from the sketch and the student's working, the valid critical values are $x = -3 + \sqrt{6}$ and $x = 1$

The solution is when the curve is above the line

So the solution in set notation is $\{x: x < -3 + \sqrt{6}\} \cup \{x: x > 1\}$

ChallengeSolving $|x^2 - 5x + 2| = |x - 3|$

$$x^2 - 5x + 2 = x - 3$$

$$x^2 - 6x + 5 = 0$$

$$(x - 5)(x - 1) = 0$$

$$x = 1, 5$$

Or

$$-(x^2 - 5x + 2) = x - 3$$

$$x^2 - 4x - 1 = 0$$

$$x = \frac{4 \pm \sqrt{20}}{2}$$

using the quadratic formula

$$x = 2 \pm \sqrt{5}$$

Solution: $x = 2 - \sqrt{5}, 1, 2 + \sqrt{5}, 5$

$$y = x^2 - 5x + 2$$

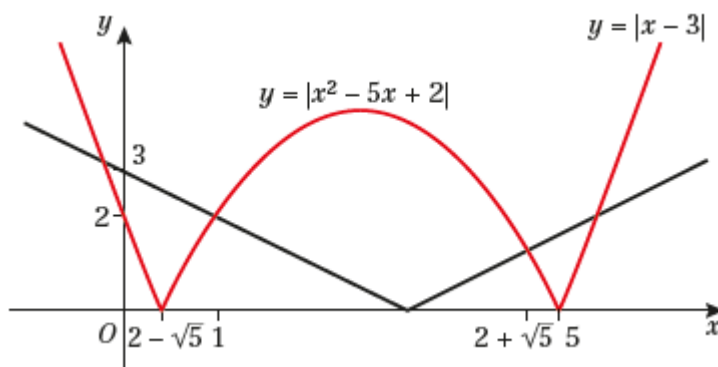
The curve is a quadratic graph with a positive x^2 coefficient. It cuts the y -axis at $(0, 2)$. It cuts the x -axis when $x = \frac{1}{2}(5 \pm \sqrt{17})$.

So the graph of $y = |x^2 - 5x + 2|$ will be the graph of $y = x^2 - 5x + 2$ but with the section of the latter curve that is below the x -axis reflected in the x -axis.

$$y = |x - 3|$$

This function has a v-shaped graph that touches the x -axis at $(3, 0)$ and crosses that y -axis at $(0, 3)$. The graph has a positive gradient for $x > 3$ and a negative gradient for $x < 3$.

So a sketch of both curves is:



The solution is when the curve is above the v-shaped graph

So the solution in set notation is $\{x: x < 2 - \sqrt{5}\} \cup \{x: 1 < x < 2 + \sqrt{5}\} \cup \{x: x > 5\}$