Vectors 1A

1 Use the results $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$, $\mathbf{i}\times\mathbf{j}=\mathbf{k}$ and $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$, and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$ and $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$

a $5\mathbf{j}\times\mathbf{k}=5(\mathbf{j}\times\mathbf{k})=5\mathbf{i}$

b $3\mathbf{i}\times\mathbf{k}=3(\mathbf{i}\times\mathbf{k})=-3\mathbf{j}$

c $\mathbf{k}\times3\mathbf{i}=3(\mathbf{k}\times\mathbf{i})=3\mathbf{j}$

d
$$\begin{aligned}3\mathbf{i}\times(9\mathbf{i}-\mathbf{j}+\mathbf{k})&=3\mathbf{i}\times9\mathbf{i}-3\mathbf{i}\times\mathbf{j}+3\mathbf{i}\times\mathbf{k}\\&=27(\mathbf{i}\times\mathbf{i})-3(\mathbf{i}\times\mathbf{j})+3(\mathbf{i}\times\mathbf{k})\\&=0-3\mathbf{k}-3\mathbf{j}=-3\mathbf{j}-3\mathbf{k}\end{aligned}$$

e
$$\begin{aligned}2\mathbf{j}\times(3\mathbf{i}+\mathbf{j}-\mathbf{k})&=2\mathbf{j}\times3\mathbf{i}+2\mathbf{j}\times\mathbf{j}-2\mathbf{j}\times\mathbf{k}\\&=6(\mathbf{j}\times\mathbf{i})+2(\mathbf{j}\times\mathbf{j})-2(\mathbf{j}\times\mathbf{k})\\&=-6\mathbf{k}-2\mathbf{i}=-2\mathbf{i}-6\mathbf{k}\end{aligned}$$

f
$$\begin{aligned}(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times2\mathbf{j}&=3\mathbf{i}\times2\mathbf{j}+\mathbf{j}\times2\mathbf{j}-\mathbf{k}\times2\mathbf{j}\\&=6(\mathbf{i}\times\mathbf{j})+2(\mathbf{j}\times\mathbf{j})-2(\mathbf{k}\times\mathbf{j})\\&=6\mathbf{k}+2\mathbf{i}=2\mathbf{i}+6\mathbf{k}\end{aligned}$$

g
$$\begin{aligned}\begin{pmatrix}5\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\3\end{pmatrix}&=(5\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+3\mathbf{k})\\&=5(\mathbf{i}\times\mathbf{i})-5(\mathbf{i}\times\mathbf{j})+15(\mathbf{i}\times\mathbf{k})+2(\mathbf{j}\times\mathbf{i})-2(\mathbf{j}\times\mathbf{j})+6(\mathbf{j}\times\mathbf{k})-(\mathbf{k}\times\mathbf{i})+(\mathbf{k}\times\mathbf{j})+3(\mathbf{k}\times\mathbf{k})\\&=-5\mathbf{k}-15\mathbf{j}-2\mathbf{k}+6\mathbf{i}-\mathbf{j}-\mathbf{i}\\&=5\mathbf{i}-16\mathbf{j}-7\mathbf{k}\\&=\begin{pmatrix}5\\-16\\-7\end{pmatrix}\end{aligned}$$

Alternatively, write the vector product as the determinant of a 3×3 matrix:

$$\begin{aligned}\begin{pmatrix}5\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\3\end{pmatrix}&=(5\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-\mathbf{j}+3\mathbf{k})\\&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&2&-1\\1&-1&3\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&-1\\-1&3\end{vmatrix}-\mathbf{j}\begin{vmatrix}5&-1\\1&3\end{vmatrix}+\mathbf{k}\begin{vmatrix}5&2\\1&-1\end{vmatrix}\\&=\mathbf{i}(6-1)-\mathbf{j}(15-(-1))+\mathbf{k}(-5-2)=5\mathbf{i}-16\mathbf{j}-7\mathbf{k}\end{aligned}$$

1 h $\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = (2\mathbf{i} - \mathbf{j} + 6\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

$$= 2(\mathbf{i}\times\mathbf{i}) - 4(\mathbf{i}\times\mathbf{j}) + 6(\mathbf{i}\times\mathbf{k}) - (\mathbf{j}\times\mathbf{i}) + 2(\mathbf{j}\times\mathbf{j}) - 3(\mathbf{j}\times\mathbf{k}) + 6(\mathbf{k}\times\mathbf{i}) - 12(\mathbf{k}\times\mathbf{j}) + 18(\mathbf{k}\times\mathbf{k})$$

$$= -4\mathbf{k} - 6\mathbf{j} + \mathbf{k} - 3\mathbf{i} + 6\mathbf{j} + 12\mathbf{i}$$

$$= 9\mathbf{i} - 3\mathbf{k}$$

$$= \begin{pmatrix} 9 \\ 0 \\ -3 \end{pmatrix}$$

Alternatively, write the vector product as the determinant of a 3×3 matrix:

$$(2\mathbf{i} - \mathbf{j} + 6\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 6 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= ((-1)\times 3 - 6\times(-2))\mathbf{i} - (2\times 3 - 6\times 1)\mathbf{j} + (2\times -2 - (-1)\times 1)\mathbf{k}$$

$$= 9\mathbf{i} - 0\mathbf{j} - 3\mathbf{k}$$

$$= 9\mathbf{i} - 3\mathbf{k}$$

i $\begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = (\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - \mathbf{k})$

$$= 2(\mathbf{i}\times\mathbf{i}) - (\mathbf{i}\times\mathbf{j}) - (\mathbf{i}\times\mathbf{k}) + 10(\mathbf{j}\times\mathbf{i}) - 5(\mathbf{j}\times\mathbf{j}) - 5(\mathbf{j}\times\mathbf{k}) - 8(\mathbf{k}\times\mathbf{i}) + 4(\mathbf{k}\times\mathbf{j}) + 4(\mathbf{k}\times\mathbf{k})$$

$$= -\mathbf{k} + \mathbf{j} - 10\mathbf{k} - 5\mathbf{i} - 8\mathbf{j} - 4\mathbf{i}$$

$$= -9\mathbf{i} - 7\mathbf{j} - 11\mathbf{k}$$

$$= \begin{pmatrix} -9 \\ -7 \\ -11 \end{pmatrix}$$

Alternatively, write the vector product as the determinant of a 3×3 matrix:

$$(\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 5 & -4 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= (5\times(-1) - (-4)\times(-1))\mathbf{i} - (1\times(-1) - (-4)\times 2)\mathbf{j} + (1\times -1 - 5\times 2)\mathbf{k}$$

$$= -9\mathbf{i} - 7\mathbf{j} - 11\mathbf{k}$$

1 j $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = (3\mathbf{i}+2\mathbf{k}) \times (\mathbf{i}-\mathbf{j}+2\mathbf{k})$

$$= 3(\mathbf{i}\times\mathbf{i}) - 3(\mathbf{i}\times\mathbf{j}) + 6(\mathbf{i}\times\mathbf{k}) + 2(\mathbf{k}\times\mathbf{i}) - 2(\mathbf{k}\times\mathbf{j}) + 4(\mathbf{k}\times\mathbf{k})$$
$$= -3\mathbf{k} - 6\mathbf{j} + 2\mathbf{j} + 2\mathbf{i}$$
$$= 2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$$
$$= \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$$

Alternatively, write the vector product as the determinant of a 3×3 matrix:

$$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 2\mathbf{i} - (6-2)\mathbf{j} - 3\mathbf{k} = 2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$$

2 Vector products can be calculated directly or by using determinants. The method using determinants is shown in these solutions.

a $\mathbf{a} = (\lambda\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \mathbf{b} = (\mathbf{i} - 3\mathbf{k})$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda & 2 & 1 \\ 1 & 0 & -3 \end{vmatrix}$$
$$= (2\times(-3) - 1\times 0)\mathbf{i} - (\lambda\times(-3) - 1\times 1)\mathbf{j} + (\lambda\times 0 - 2\times 1)\mathbf{k}$$
$$= -6\mathbf{i} + (3\lambda+1)\mathbf{j} - 2\mathbf{k}$$

b $\mathbf{a} = (2\mathbf{i} - \mathbf{j} + 7\mathbf{k}), \mathbf{b} = (\mathbf{i} - \lambda\mathbf{j} + 3\mathbf{k})$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 7 \\ 1 & -\lambda & 3 \end{vmatrix}$$
$$= (-1\times 3 - 7\times(-\lambda))\mathbf{i} - (2\times 3 - 7\times 1)\mathbf{j} + (2\times(-\lambda) - (-1)\times 1)\mathbf{k}$$
$$= (7\lambda - 3)\mathbf{i} + \mathbf{j} + (1-2\lambda)\mathbf{k}$$

3 Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = (4\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 4 & 1 & 3 \end{vmatrix}$$
$$= -3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$$

$\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**.

So to obtain the unit vector, find the magnitude of $\mathbf{a} \times \mathbf{b}$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-3)^2 + (-6)^2 + 6^2}$$
$$= \sqrt{9 + 36 + 36} = \sqrt{81} = 9$$

So a unit vector perpendicular to both **a** and **b** is

$$\frac{1}{9}(\mathbf{a} \times \mathbf{b}) = \frac{1}{9}(-3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}) = -\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

Multiplying this vector by –1 will give another unit vector that is perpendicular to **a** and **b**, so another possible answer is $\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

4 Let $\mathbf{a} = 4\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{j} - \sqrt{2}\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{vmatrix}$$
$$= -\mathbf{i} + 4\sqrt{2}\mathbf{j} + 4\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-1)^2 + (4\sqrt{2})^2 + 4^2}$$
$$= \sqrt{1 + 32 + 16} = \sqrt{49} = 7$$

So $\frac{1}{7}(-\mathbf{i} + 4\sqrt{2}\mathbf{j} + 4\mathbf{k})$ is a unit vector, which is perpendicular to $4\mathbf{i} + \mathbf{k}$ and to $\mathbf{j} - \sqrt{2}\mathbf{k}$

5 Let $\mathbf{a} = \mathbf{i} - \mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 3 & 4 & -6 \end{vmatrix}$$
$$= 6\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{6^2 + 6^2 + 7^2}$$
$$= \sqrt{36 + 36 + 49} = \sqrt{121} = 11$$

So $\frac{1}{11}(6\mathbf{i} + 6\mathbf{j} + 7\mathbf{k})$ is the required unit vector.

6 Let $\mathbf{a} = \mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 9\mathbf{j} + 8\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 4 \\ 5 & 9 & 8 \end{vmatrix} = 12\mathbf{i} + 12\mathbf{j} - 21\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{12^2 + 12^2 + (-21)^2}$$
$$= \sqrt{144 + 144 + 441} = \sqrt{729} = 27$$

So the required unit vector is $\frac{1}{27}(12\mathbf{i} + 12\mathbf{j} - 21\mathbf{k}) = \frac{1}{9}(4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) = \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{7}{9} \end{pmatrix}$

7 Let $\mathbf{a} = 4\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \sqrt{2}\mathbf{j} + \mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 0 & \sqrt{2} & 1 \end{vmatrix} = -\sqrt{2}\mathbf{i} - 4\mathbf{j} + 4\sqrt{2}\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-\sqrt{2})^2 + (-4^2) + (4\sqrt{2})^2}$$
$$= \sqrt{(2 + 16 + 32)} = \sqrt{50} = 5\sqrt{2}$$

So $\frac{1}{\sqrt{2}}(\mathbf{a} \times \mathbf{b})$ has magnitude 5 and is perpendicular to **a** and **b**.

So the required vector is $\frac{1}{\sqrt{2}}(-\sqrt{2}\mathbf{i} - 4\mathbf{j} + 4\sqrt{2}\mathbf{k}) = -\mathbf{i} - 2\sqrt{2}\mathbf{j} + 4\mathbf{k} = \begin{pmatrix} -1 \\ -2\sqrt{2} \\ 4 \end{pmatrix}$

8 Let $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 0\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = -2\mathbf{j} - 2\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-2)^2 + (-2)^2}$$
$$= \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2} = 2.83 \text{ (3 s.f.)}$$

9 **a** $\mathbf{a}.\mathbf{b} = (-1) \times 5 + 2 \times (-2) + (-5) \times 1 = -5 - 4 - 5 = -14$

b $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -5 \\ 5 & -2 & 1 \end{vmatrix} = -8\mathbf{i} - 24\mathbf{j} - 8\mathbf{k}$

9 c $|\mathbf{a}\times\mathbf{b}| = \sqrt{(-8)^2+(-24)^2+(-8)^2} = 8\sqrt{(-1)^2+(-3)^2+(-1)^2} = 8\sqrt{11}$

Unit vector in direction $\mathbf{a}\times\mathbf{b}$ is $\dfrac{1}{8\sqrt{11}}(-8\mathbf{i}-24\mathbf{j}-8\mathbf{k}) = \dfrac{1}{\sqrt{11}}(-\mathbf{i}-3\mathbf{j}-\mathbf{k})$

10 Use $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$ for these problems.

a $|\mathbf{a}| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$

$|\mathbf{b}| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 0 \\ 2 & 2 & 1 \end{vmatrix} = -4\mathbf{i}-3\mathbf{j}+14\mathbf{k}$$

$|\mathbf{a}\times\mathbf{b}| = \sqrt{(-4)^2+(-3)^2+14^2} = \sqrt{221}$

Let θ be the angle between **a** and **b** then

$$\sin\theta = \frac{\sqrt{221}}{5\times 3} = \frac{\sqrt{221}}{15}$$

b $|\mathbf{a}| = \sqrt{1^2+2^2} = \sqrt{5}$

$|\mathbf{b}| = \sqrt{5^2+4^2+(-2)^2} = \sqrt{45} = 3\sqrt{5}$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 5 & 4 & -2 \end{vmatrix} = -10\mathbf{i}+10\mathbf{j}-5\mathbf{k}$$

$|\mathbf{a}\times\mathbf{b}| = \sqrt{(-10)^2+10^2+(-5)^2} = \sqrt{225} = 15$

Let θ be the angle between **a** and **b** then

$$\sin\theta = \frac{15}{\sqrt{5}\times 3\sqrt{5}} = \frac{15}{15} = 1$$

c $|\mathbf{a}| = \sqrt{5^2+2^2+2^2} = \sqrt{33}$ $\quad |\mathbf{b}| = \sqrt{4^2+4^2+1^2} = \sqrt{33}$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & 2 \\ 4 & 4 & 1 \end{vmatrix} = -6\mathbf{i}+3\mathbf{j}+12\mathbf{k} \qquad |\mathbf{a}\times\mathbf{b}| = \sqrt{(-6)^2+3^2+12^2} = \sqrt{189} = 3\sqrt{21}$$

Let θ be the angle between **a** and **b** then

$$\sin\theta = \frac{3\sqrt{21}}{\sqrt{33}\times\sqrt{33}} = \frac{3\sqrt{21}}{33} = \frac{\sqrt{21}}{11}$$

11 The direction of line l_1 is $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$
The direction of the l_2 is $2\mathbf{i}-\mathbf{j}+\mathbf{k}$

A vector perpendicular to both l_1 and l_2 is in the direction:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 5\mathbf{i}+5\mathbf{j}-5\mathbf{k}$$

So any multiple of $(\mathbf{i}+\mathbf{j}-\mathbf{k})$ is perpendicular to lines l_1 and l_2

12 Calculating the vector product using the determinant gives:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & u & v \end{vmatrix}$$
$$= (3v+u)\mathbf{i}-(v+2)\mathbf{j}+(u-6)\mathbf{k}$$

As $\mathbf{a}\times\mathbf{b} = w\mathbf{i}-6\mathbf{j}-7\mathbf{k}$, equating **i**, **j** and **k** components gives:

$3v+u=w \quad (1)$

$v+2=6 \quad (2)$

$u-6=-7 \quad (3)$

From equation (2): $v=4$
From equation (3): $u=-1$
Substituting for v and u in equation (1): w = 12 – 1 i.e. w = 11

So solution is $u=-1,\ v=4$ and $w=11$

13 a Calculating the vector product using the determinant gives:

$$\mathbf{q}\times\mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ a & -1 & 4 \end{vmatrix}$$
$$= 3\mathbf{i}-a\mathbf{j}-a\mathbf{k}$$

As $\mathbf{q}\times\mathbf{p} = 3\mathbf{i}-\mathbf{j}+b\mathbf{k}$, equating components of **i**, **j** and **k** gives:
$a=1$, from **j** component.
$-a=b$, from **k** component, so $b=-1$

Solution is $a=1$ and $b=-1$

b $\mathbf{p.q} = a\times 0+(-1)\times 1+4\times(-1)=-5$
$|\mathbf{p}|=\sqrt{a^2+(-1)^2+4^2}=\sqrt{18}$ as $a=1$ $\quad |\mathbf{q}|=\sqrt{1^2+(-1)^2}=\sqrt{2}$

From definition of scalar product $\cos\theta=\dfrac{\mathbf{p.q}}{|\mathbf{p}||\mathbf{q}|}\Rightarrow\cos\theta=\dfrac{-5}{\sqrt{18}\sqrt{2}}=\dfrac{-5}{\sqrt{36}}=-\dfrac{5}{6}$

This gives the obtuse angle between the vectors.

The cosine of the corresponding acute angle is $\dfrac{5}{6}$

14 Given $\mathbf{a}\times\mathbf{b}=0$ and $\mathbf{a}\neq 0$ and $\mathbf{b}\neq 0$, this implies that **a** is parallel to **b**, i.e. $\mathbf{a}=c\mathbf{b}$ where c is a scalar constant. So:

$$\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}3c\\\lambda c\\\mu c\end{pmatrix}$$

Comparing each term of the matrices gives:

$$3c=2\Rightarrow c=\frac{2}{3}$$

$$1=\lambda c=\frac{2}{3}\lambda\Rightarrow\lambda=\frac{3}{2}$$

$$-1=\mu c=\frac{2}{3}\mu\Rightarrow\mu=-\frac{3}{2}$$

An alternative method is to find the vector product:

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&-1\\3&\lambda&\mu\end{vmatrix}=(\mu+\lambda)\mathbf{i}-(2\mu+3)\mathbf{j}+(2\lambda-3)\mathbf{k}$$

As $\mathbf{a}\times\mathbf{b}=0\Rightarrow\mu+\lambda=0,\ 2\mu+3=0,\ 2\lambda-3=0$

$\Rightarrow\lambda=\frac{3}{2}$ and $\mu=-\frac{3}{2}$

15 $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ (1)

Multiply equation (1) first by **a** and then by **b**.
First taking the vector product of **a** and equation (1)

$\mathbf{a}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{a}\times 0$

$\Rightarrow\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=0$

$\Rightarrow\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=0$ as $\mathbf{a}\times\mathbf{a}=0$

$\Rightarrow\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}=0$ as $\mathbf{a}\times\mathbf{c}=-\mathbf{c}\times\mathbf{a}$

$\Rightarrow\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$

Taking the vector product of **b** and equation (1)

$\mathbf{b}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{b}\times 0$

$\Rightarrow\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}=0$

$\Rightarrow\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}=0$ as $\mathbf{b}\times\mathbf{b}=0$

$\Rightarrow-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}=0$ as $\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$

$\Rightarrow\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}$

So $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$

Challenge

To show that $\mathbf{a}$ is parallel to $\mathbf{b}+\mathbf{c}$, show that $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=0$

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{c} \qquad \text{as } \mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$$
$$=-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{c}=0 \qquad \text{as } \mathbf{c}\times\mathbf{a}=-\mathbf{a}\times\mathbf{c}$$

As $\mathbf{a}$ is non-zero, this implies that $\mathbf{a}$ is parallel to $\mathbf{b}+\mathbf{c}$