The binomial expansion, Mixed Exercise 8

1 a ${}^{16-1}C_{4-1} = {}^{15}C_3 = 455$

${}^{16-1}C_{5-1} = {}^{15}C_4 = 1365$

b The coefficients are 1, 15, 105, 455, 1365, ...

x^3 term of $(1+2x)^{15} = 455(1)^{12}(2x)^3 = 3640x^3$

Coefficient = 3640

2 $\binom{45}{17} = \frac{45!}{17!a!}$

$\binom{45}{17} = \frac{45!}{17!28!}$

$a = 28$

3 a When $n = 5$ and $p = 0.5$,

$\binom{20}{n} p^n (1-p)^{20-n} = \binom{20}{5} 0.5^5 (1-0.5)^{20-15}$

$= 0.0148$ (to 3 s.f.)

b When $n = 0$ and $p = 0.7$,

$\binom{20}{n} p^n (1-p)^{20-n} = \binom{20}{0} 0.7^0 (1-0.7)^{20}$

$= 0.000\,000\,000\,034\,9$ (to 3 s.f.)

c When $n = 13$ and $p = 0.6$,

$\binom{20}{n} p^n (1-p)^{20-n} = \binom{20}{13} 0.6^{13} (1-0.6)^7$

$= 0.166$ (to 3 s.f.)

4

$$\left(1-\frac{3x}{2}\right)^p = 1+\binom{p}{1}1^{p-1}\left(-\frac{3x}{2}\right)+\binom{p}{2}1^{p-2}\left(-\frac{3x}{2}\right)+\binom{p}{3}1^{p-3}\left(-\frac{3x}{2}\right)+\ldots$$

$$= 1+p\left(-\frac{3x}{2}\right)+\frac{p(p-1)}{2!}\left(-\frac{3x}{2}\right)^2+\frac{p(p-1)(p-2)}{3!}\left(-\frac{3x}{2}\right)^3+\ldots$$

a Coefficient of x is $-\frac{3p}{2}$.

$-\frac{3p}{2} = -24$

$p = 16$

b Coefficient of $x^2 = \frac{p(p-1)}{2}\times\frac{9}{4} = \frac{16\times15}{2}\times\frac{9}{4} = 270$

c Coefficient of $x^3 = -\frac{p(p-1)(p-2)}{3!}\times\frac{27}{8} = -\frac{16\times15\times14}{3\times2}\times\frac{27}{8} = -1890$

5 $(2-x)^{13} = 2^{13} + \binom{13}{1}2^{12}(-x) + \binom{13}{2}2^{11}(-x)^2 + \ldots$

$= 8192 + 13 \times (-4096x) + 78 \times 2048x^2 + \ldots$

$= 8192 - 53\,248x + 159\,744x^2 + \ldots$

$= A + Bx + Cx^2 + \ldots$

So $A = 8192$, $B = -53\,248$, $C = 159\,744$

6 a $(1-2x)^{10} = 1 + \binom{10}{1}1^9(-2x) + \binom{10}{2}1^8(-2x)^2 + \binom{10}{3}1^7(-2x)^3 + \ldots$

$= 1 + 10 \times (-2x) + 45 \times (-2x)^2 + 120 \times (-2x)^3 + \ldots$

$= 1 - 20x + 180x^2 - 960x^3 + \ldots$

b We need $(1-2x) = 0.98$

$2x = 0.02$

$x = 0.01$

Substituting $x = 0.01$ into the expansion for $(1-2x)^{10}$:

$0.98^{10} \approx 1 - 20 \times 0.01 + 180 \times 0.01^2 - 960 \times 0.01^3$

$= 0.81\,704 + \ldots$

7 a $(2-3x)^{10} = 2^{10} + \binom{10}{1}2^9(-3x) + \binom{10}{2}2^8(-3x)^2 + \binom{10}{3}2^7(-3x)^3 + \ldots$

$= 1024 + 10 \times (-1536x) + 45 \times 2304x^2 + 120 \times (-3456x^3) + \ldots$

$= 1024 - 15\,360x + 103\,680x^2 - 414\,720x^3 + \ldots$

b We require $(2-3x) = 1.97$

$3x = 0.03$

$x = 0.01$

Substituting $x = 0.01$ in the expansion for $(2-3x)^{10}$:

$1.97^{10} \approx 1024 - 15\,360 \times 0.01 + 103\,680 \times 0.01^2 - 414\,720 \times 0.01^3$

$= 1024 - 153.6 + 10.368 - 0.414\,72$

$= 880.35$ (to 2 d.p.)

8 a $(3+2x)^4 = 3^4 + \binom{4}{1}3^3(2x) + \binom{4}{2}3^2(2x)^2 + \binom{4}{3}3(2x)^3 + (2x)^4$

$= 3^4 + 4 \times 54x + 6 \times 36x^2 + 4 \times 24x^3 + 16x^4$

$= 81 + 216x + 216x^2 + 96x^3 + 16x^4$

b Substituting $x = -x$:

$(3-2x)^4 = 81 + 216(-x) + 216(-x)^2 + 96(-x)^3 + 16(-x)^4$

$= 81 - 216x + 216x^2 - 96x^3 + 16x$

c Using parts **a** and **b**:

$$\begin{aligned}(3+2x)^4 + (3-2x)^4 = &\ \ 81 + 216x + 216x^2 + 96x^3 + 16x^4\\ &+ 81 - 216x + 216x^2 - 96x^3 + 16x^4\\ \hline &\ \ 162 \qquad\quad + 432x^2 \qquad\quad + 32x^4\end{aligned}$$

Substituting $x = \sqrt{2}$ into both sides of this expansion gives:

$\left(3+2\sqrt{2}\right)^4 + \left(3-2\sqrt{2}\right)^4 = 164 + 432\left(\sqrt{2}\right)^2 + 32\left(\sqrt{2}\right)^4$

8 c $\left(3+2\sqrt{2}\right)^4+\left(3-2\sqrt{2}\right)^4 = 162 + 432\times 2 + 32\times 4$

$= 1154$

9 $\left(1+\frac{x}{2}\right)^n \ldots = 1+\binom{n}{1}1^{n-1}\left(\frac{x}{2}\right)+\binom{n}{2}1^{n-2}\left(\frac{x}{2}\right)^2+\binom{n}{3}1^{n-3}\left(\frac{x}{2}\right)^3+\ldots$

$= 1+n\left(\frac{x}{2}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{2}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{2}\right)^3+\frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{x}{2}\right)^4+\ldots$

a x^2 term $= \frac{n(n-1)}{2!\times 4}x^2$

$\frac{n(n-1)}{2!\times 4} = 7$

$n(n-1) = 56$

$n^2 - n - 56 = 0$

$(n-8)(n+7) = 0$

n is a positive integer, so $n = 8$

b Coefficient of $x^4 = \frac{n(n-1)(n-2)(n-3)}{4!}\times\frac{1}{2^4}$

$= \frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1}\times\frac{1}{16}$

$= \frac{35}{8}$

10 a $(3+10x)^4 = 3^4 + \binom{4}{1}3^3(10x) + \binom{4}{2}3^2(10x)^2 + \binom{4}{3}3(10x)^3 + (10x)^4$

$= 3^4 + 4\times 270x + 6\times 900x^2 + 4\times 3000x^3 + 10\,000x^4$

$= 81 + 1080x + 5400x^2 + 12\,000x^3 + 10\,000x^4$

b We require $(3 + 10x) = 1003$

$10x = 1000$

$x = 100$

Substituting $x = 100$ in the expansion of $(3 + 10x)^4$:

$1003^4 = 81 + 1080\times 100 + 5400\times 100^2 + 12\,000\times 100^3 + 10\,000\times 100^4$

$= 81 + 108\,000 + 54\,000\,000 + 12\,000\,000\,000 + 1\,000\,000\,000\,000$

$$\begin{array}{r} 1\,000\,000\,000\,000 \\ 12\,000\,000\,000 \\ 54\,000\,000 \\ 108\,000 \\ 81 \\ \hline 1\,012\,054\,108\,081 \end{array}$$

$1003^4 = 1\,012\,054\,108\,081$

11 a $(1 + 2x)^{12}$

$= 1^{12} + \binom{12}{1}1^{11}(2x) + \binom{12}{2}1^{10}(2x)^2 + \binom{12}{3}1^9(2x)^3 + \ldots$

$= 1 + 12 \times 2x + 66 \times 4x^2 + 220 \times 8x^3 + \ldots$

$= 1 + 24x + 264x^2 + 1760x^3 + \ldots$

b We want $(1 + 2x) = 1.02$

$2x = 0.02$

$x = 0.01$

Substituting $x = 0.01$ in the expansion for $(1 + 2x)^{12}$:

$1.02^{12} \approx 1 + 24 \times 0.01 + 264 \times 0.01^2 + 1760 \times 0.01^3$

$= 1.268\,16$

c Using a calculator:

$1.02^{12} = 1.268\,241\,795$

d Error $= \dfrac{1.268\ 241\ 795\ -\ 1.268\ 16}{1.268\ 241\ 795} \times 100 = 0.006\,45\%$

12 $\left(x - \frac{1}{x}\right)^5$ has coefficients and terms

1	5	10	10	5	1
x^2	$x^4\left(-\frac{1}{x}\right)$	$x^3\left(-\frac{1}{x}\right)^2$	$x^2\left(-\frac{1}{x}\right)^3$	$x\left(-\frac{1}{x}\right)^4$	$\left(-\frac{1}{x}\right)^5$

Putting these together gives:

$$\left(x - \frac{1}{x}\right)^5 = 1x^5 + 5x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + 1\left(-\frac{1}{x}\right)^5$$

$$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

13 a $(2k + x)^n = (2k)^n + \binom{n}{1}(2k)^{n-1}x + \binom{n}{2}(2k)^{n-2}x^2 + \binom{n}{3}(2k)^{n-3}x^3 + \ldots$

Coefficient of x^2 = coefficient of x^3

$$\binom{n}{2}(2k)^{n-2} = \binom{n}{3}(2k)^{n-3}$$

$$\frac{\cancel{n}!}{(n-2)!2!}(2k)^{n-2} = \frac{\cancel{n}!}{(n-3)!3!}(2k)^{n-3}$$

$$\frac{(2k)^{n-2}}{(2k)^{n-3}} = \frac{(n-2)!2!}{(n-3)!3!}$$

$$(2k)^1 = \frac{(n-2)!2!}{(n-3)!3!}$$

$$= \frac{(n-2)\times(n-3)!2!}{(n-3)!3!}$$

13 a $2k = \dfrac{(n-2)\times \cancel{2}}{\cancel{6}_{\,3}}$

$3 \times 2k = n - 2$
$6k = n - 2$
$n = 6k + 2$

b If $k = \frac{2}{3}$ then $n = 6 \times \frac{2}{3} + 2 = 6$

$$\left(2\times\frac{2}{3}+x\right)^6 = \left(\frac{4}{3}+x\right)^6$$
$$= \left(\frac{4}{3}\right)^6 + \binom{6}{1}\left(\frac{4}{3}\right)^5 x + \binom{6}{2}\left(\frac{4}{3}\right)^4 x^2 + \binom{6}{3}\left(\frac{4}{3}\right)^3 x^3 + \ldots$$
$$= \frac{4096}{729} + \frac{2048}{81}x + \frac{1280}{27}x^2 + \frac{1280}{27}x^3 + \ldots$$

14 a $(2 + x)^6 = 2^6 + \binom{6}{1}2^5x + \binom{6}{2}2^4x^2 + \binom{6}{3}2^3x^3 + \binom{6}{4}2^2x^4 + \binom{6}{5}2x^5 + x^6$
$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

b With $x = \sqrt{3}$

$$\left(2+\sqrt{3}\right)^6 = 64+192\sqrt{3}+240\left(\sqrt{3}\right)^2+160\left(\sqrt{3}\right)^3+60\left(\sqrt{3}\right)^4+12\left(\sqrt{3}\right)^5+\left(\sqrt{3}\right)^6 \quad \textbf{(1)}$$

With $x = -\sqrt{3}$

$$\left(2-\sqrt{3}\right)^6 = 64+192\left(-\sqrt{3}\right)+240\left(-\sqrt{3}\right)^2+160\left(-\sqrt{3}\right)^3+60\left(-\sqrt{3}\right)^4+12\left(-\sqrt{3}\right)^5+\left(-\sqrt{3}\right)^6$$
$$\left(2-\sqrt{3}\right)^6 = 64-192\sqrt{3}+240\left(\sqrt{3}\right)^2-160\left(\sqrt{3}\right)^3+60\left(\sqrt{3}\right)^4-12\left(\sqrt{3}\right)^5+\left(\sqrt{3}\right)^6 \quad \textbf{(2)}$$

(**1**)− (**2**) gives:

$$\left(2+\sqrt{3}\right)^6-\left(2-\sqrt{3}\right)^6 = 384\sqrt{3}+320\left(\sqrt{3}\right)^3+24\left(\sqrt{3}\right)^5$$
$$= 384\sqrt{3}+320\times 3\sqrt{3}+24\times 3\times 3\sqrt{3}$$
$$= 384\sqrt{3}+960\sqrt{3}+216\sqrt{3}$$
$$= 1560\sqrt{3}$$

Hence $k = 1560$

15 a The term in x^2 of $(2 + kx)^8$ is

$\binom{8}{2}2^6(kx)^2 = 28 \times 64k^2x^2 = 1792k^2x^2$

$1792k^2 = 2800$
$k^2 = 1.5625$
$k = \pm 1.25$
k is positive, so $k = 1.25$

15 b Term in x^3 of $(2 + kx)^8$ is

$\binom{8}{3}2^5(kx)^3 = 56 \times 32k^3x^3 = 1792k^3x^3$

Coefficient of x^3 term is $1792k^3 = 1792 \times 1.25^3 = 3500$

16 a $(2 + x)^5$ has coefficients and terms

1	5	10	10	5	1
2^5	2^4x	2^3x^2	2^2x^3	$2x^4$	x^5

Putting these together gives:

$(2 + x)^5 = 1 \times 2^5 + 5 \times 2^4x + 10 \times 2^3x^2 + 10 \times 2^2x^3 + 5 \times 2x^4 + 1 \times x^5$

$(2 + x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

Substituting $x = -x$:

$(2 - x)^5 = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

Adding:

$(2 + x)^5 + (2 - x)^5 = 64 + 160x^2 + 20x^4$

So $A = 64$, $B = 160$ and $C = 20$

b $(2 + x)^5 + (2 - x)^5 = 349$

$64 + 160x^2 + 20x^4 = 349$

$20x^4 + 160x^2 - 285 = 0$

$4x^4 + 32x^2 - 57 = 0$

Substituting $y = x^2$:

$4y^2 + 32y - 57 = 0$

$(2y - 3)(2y + 19) = 0$

$y = \frac{3}{2}, -\frac{19}{2}$

But $y = x^2$, so $x^2 = \frac{3}{2} \Rightarrow x = \pm\sqrt{\frac{3}{2}}$

17 a x^3 term $= \binom{5}{3}2^2(px)^3 = 10 \times 4p^3x^3 = 40p^3x^3$

$40p^3 = 135$

$p^3 = 3.375$

$p = 1.5$

b x^4 term $= \binom{5}{4}2(px)^4$

$= 5 \times 2p^4x^4$

$= 5 \times 2(1.5)^4x^4$

$= 50.625x^4$

Coefficient = 50.625

18 $\left(\frac{x^2}{2}-\frac{2}{x}\right)^9$

Constant term = $\binom{9}{6}\left(\frac{x^2}{2}\right)^3\left(-\frac{2}{x}\right)^6$

$= 84\times\left(\frac{x^6}{8}\right)\times\left(\frac{64}{x^6}\right)$

$= 672$

19 a $(2+px)^7 = 2^7+\binom{7}{1}2^6(px)^1+\binom{7}{2}2^5(px)^2+\ldots$

$= 128 + 448px + 672p^2x^2 + \ldots$

b $448p = 2240 \Rightarrow p = 5$

$672p^2 = q$

$672 \times 5^2 = q$

$q = 16\,800$

$p = 5$ and $q = 16\,800$

20 a $(1-px)^{12} = 1^{12}+\binom{12}{1}1^{11}(-px)+\binom{12}{2}1^{10}(-px)^2+\ldots$

$= 1 - 12px + 66p^2x^2 + \ldots$

b $-12p = q$ and $66p^2 = 6q$

$11p^2 = q$

Substituting $-12p = q$ into $11p^2 = q$ gives:

$11p^2 = -12p$

$11p^2 + 12p = 0$

$p(11p + 12) = 0$

$p = 0$ or $-\frac{12}{11} = -1\frac{1}{11}$

p is a non-zero constant, so $p = -1\frac{1}{11}$

$q = -12 \times -\frac{12}{11} = \frac{144}{11} = 13\frac{1}{11}$

$p = -1\frac{1}{11}$ and $q = 13\frac{1}{11}$

21 a $\left(2+\frac{x}{2}\right)^7 = 2^7+\binom{7}{1}2^6\left(\frac{x}{2}\right)+\binom{7}{2}2^5\left(\frac{x}{2}\right)^2+\ldots$

$= 128 + 224x + 168x^2 + \ldots$

b We want $\left(2+\frac{x}{2}\right) = 2.05$

$\frac{x}{2} = 0.05$

$x = 0.1$

Substitute $x = 0.1$ into the expansion for $\left(2+\frac{x}{2}\right)^7$ assuming the terms after x^2 are negligible.

22 $(4 + kx)^5$

x^3 term $= \binom{5}{3} 4^2 (kx)^3$

$= 10 \times 16 \times k^3x^3$

$= 160k^3x^3$

$160k^3 = 20$

$k^3 = \frac{1}{8}$

$k = \frac{1}{2}$

Challenge

a $(3 + x)^5 = 3^5 + \binom{5}{1}3^4 x + \binom{5}{2}3^3 x^2 + \ldots$

$= 243 + 405x + 270x^2 + \ldots$

$(2 - px)(3 + x)^5 = (2 - px)(243 + 405x + 270x^2 + \ldots)$

$= 486 + 810x + 540x^2 - 243px - 405px^2 + \ldots$

x^2 term $= (540 - 405p)x^2$

$540 - 405p = 0$

$405p = 540$

$p = \frac{540}{405} = \frac{4}{3}$

b $(1 + 2x)^8 = 1^8 + \binom{8}{1}1^7(2x) + \binom{8}{2}1^6(2x)^2 + \ldots$

$= 1 + 16x + 112x^2 + \ldots$

$(2 - 5x)^7 = 2^7 + \binom{7}{1}2^6(-5x) + \binom{7}{2}2^5(-5x)^2 + \ldots$

$= 128 - 2240x + 16\,800x^2 + \ldots$

The x^2 term in the expansion of $(1 + 2x)^8(2 - 5x)^7$

$= 1 \times 16\,800x^2 + 16x \times (-2240x) + 128 \times 112x^2$

$= -4704x^2$

The coefficient of the x^2 term is -4704.