

Differentiation 9E

1 a Let $y = \frac{5x}{x+1}$

Let $u = 5x$ and $v = x+1$

Then $\frac{du}{dx} = 5$ and $\frac{dv}{dx} = 1$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x+1) \times 5 - 5x \times 1}{(x+1)^2} = \frac{5}{(x+1)^2}$$

b Let $y = \frac{2x}{3x-2}$

Let $u = 2x$ and $v = 3x-2$

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 3$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(3x-2) \times 2 - 2x \times 3}{(3x-2)^2} \\ &= \frac{6x-4-6x}{(3x-2)^2} = -\frac{4}{(3x-2)^2} \end{aligned}$$

c Let $y = \frac{x+3}{2x+1}$

Let $u = x+3$ and $v = 2x+1$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x+1) \times 1 - (x+3) \times 2}{(2x+1)^2} \\ &= \frac{2x+1-2x-6}{(2x+1)^2} = -\frac{5}{(2x+1)^2} \end{aligned}$$

d Let $y = \frac{3x^2}{(2x-1)^2}$

Let $u = 3x^2$ and $v = (2x-1)^2$

Then $\frac{du}{dx} = 6x$ and $\frac{dv}{dx} = 4(2x-1)$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x-1)^2 \times 6x - 3x^2 \times 4(2x-1)}{(2x-1)^4} \\ &= \frac{6x(2x-1)((2x-1)-2x)}{(2x-1)^4} \\ &= \frac{-6x(2x-1)}{(2x-1)^4} = -\frac{6x}{(2x-1)^3} \end{aligned}$$

e Let $y = \frac{6x}{(5x+3)^{\frac{1}{2}}}$

Let $u = 6x$ and $v = (5x+3)^{\frac{1}{2}}$

Then $\frac{du}{dx} = 6$ and $\frac{dv}{dx} = \frac{5}{2}(5x+3)^{-\frac{1}{2}}$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(5x+3)^{\frac{1}{2}} \times 6 - 6x \times \frac{5}{2}(5x+3)^{-\frac{1}{2}}}{\left((5x+3)^{\frac{1}{2}}\right)^2} \\ &= \frac{3(5x+3)^{-\frac{1}{2}}(2(5x+3)-5x)}{(5x+3)} \\ &= \frac{3(5x+3)^{-\frac{1}{2}}(10x+6-5x)}{(5x+3)} = \frac{3(5x+6)}{(5x+3)^{\frac{3}{2}}} \end{aligned}$$

2 a Let $y = \frac{e^{4x}}{\cos x}$

Let $u = e^{4x}$ and $v = \cos x$

$$\frac{du}{dx} = 4e^{4x} \text{ and } \frac{dv}{dx} = -\sin x$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{4e^{4x} \cos x - e^{4x}(-\sin x)}{\cos^2 x} \\ &= \frac{e^{4x}(4 \cos x + \sin x)}{\cos^2 x} \end{aligned}$$

b Let $y = \frac{\ln x}{x+1}$

Let $u = \ln x$ and $v = x+1$

$$\frac{du}{dx} = \frac{1}{x} \text{ and } \frac{dv}{dx} = 1$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\frac{(x+1)}{x} - \ln x}{(x+1)^2} = \frac{1}{x(x+1)} - \frac{\ln x}{(x+1)^2}$$

c Let $y = \frac{e^{-2x} + e^{2x}}{\ln x}$

Let $u = e^{-2x} + e^{2x}$ and $v = \ln x$

$$\frac{du}{dx} = -2e^{-2x} + 2e^{2x} \text{ and } \frac{dv}{dx} = \frac{1}{x}$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\ln x (2e^{2x} - e^{-2x}) - \frac{e^{-2x} + e^{2x}}{x}}{(\ln x)^2} \\ &= \frac{2x \ln x (e^{2x} - e^{-2x}) - (e^{-2x} + e^{2x})}{x(\ln x)^2} \\ &= \frac{e^{-2x} (2x(e^{4x} - 1) \ln x - e^{4x} - 1)}{x(\ln x)^2} \end{aligned}$$

d Let $y = \frac{(e^x + 3)^3}{\cos x}$

Let $u = (e^x + 3)^3$ and $v = \cos x$

$$\frac{du}{dx} = 3e^x(e^x + 3)^2 \text{ and } \frac{dv}{dx} = -\sin x$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3e^x(e^x + 3)^2 \cos x - (-\sin x)(e^x + 3)^3}{\cos^2 x} \\ &= \frac{(e^x + 3)^2 (3e^x \cos x + (e^x + 3) \sin x)}{\cos^2 x} \end{aligned}$$

e Let $y = \frac{\sin^2 x}{\ln x}$

Let $u = \sin^2 x$ and $v = \ln x$

$$\frac{du}{dx} = 2 \sin x \cos x \text{ and } \frac{dv}{dx} = \frac{1}{x}$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\ln x (2 \sin x \cos x) - \frac{1}{x} \sin^2 x}{(\ln x)^2} \\ &= \frac{2 \sin x \cos x}{\ln x} - \frac{\sin^2 x}{x(\ln x)^2} \end{aligned}$$

3 $y = \frac{x}{3x+1}$

Let $u = x$ and $v = 3x+1$

$$\frac{du}{dx} = 1 \text{ and } \frac{dv}{dx} = 3$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(3x+1) - 3x}{(3x+1)^2} = \frac{1}{(3x+1)^2}$$

At the point $(1, \frac{1}{4})$, $x = 1$

so $\frac{dy}{dx} = \frac{1}{4^2} = \frac{1}{16}$

4 $y = \frac{x+3}{(2x+1)^{\frac{1}{2}}}$

Let $u = x+3$ and $v = (2x+1)^{\frac{1}{2}}$

$$\frac{du}{dx} = 1 \text{ and } \frac{dv}{dx} = (2x+1)^{-\frac{1}{2}}$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(2x+1)^{\frac{1}{2}} - (x+3)(2x+1)^{-\frac{1}{2}}}{2x+1}$$

At the point $(12, 3)$, $x = 12$

so $\frac{dy}{dx} = \frac{25^{\frac{1}{2}} - (15 \times 25^{-\frac{1}{2}})}{25}$
 $= \frac{5 - 15 \times \frac{1}{5}}{25} = \frac{2}{25}$

5 $y = \frac{e^{2x+3}}{x}$

Let $u = e^{2x+3}$ and $v = x$

$$\frac{du}{dx} = 2e^{2x+3} \text{ and } \frac{dv}{dx} = 1$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{2xe^{2x+3} - e^{2x+3}}{x^2} = \frac{e^{2x+3}(2x-1)}{x^2}$$

At stationary points $\frac{dy}{dx} = 0$

so $2x-1=0$

$x = 0.5$ and $y = 2e^4$

There is one stationary point at $(0.5, 2e^4)$.

6 $y = \frac{e^{\frac{1}{3}x}}{x}$

Let $u = e^{\frac{1}{3}x}$ and $v = x$

$$\frac{du}{dx} = \frac{1}{3}e^{\frac{1}{3}x} \text{ and } \frac{dv}{dx} = 1$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\frac{x}{3}e^{\frac{1}{3}x} - e^{\frac{1}{3}x}}{x^2} = \frac{e^{\frac{1}{3}x} \left(\frac{x}{3} - 1 \right)}{x^2}$$

At the point $(3, \frac{1}{3}e)$, $x = 3$ so $\frac{dy}{dx} = 0$

Equation of tangent is

$$y - \frac{1}{3}e = 0(x-3)$$

i.e. $y = \frac{1}{3}e$

7 $y = \frac{\ln x}{\sin 3x}$

Let $u = \ln x$ and $v = \sin 3x$

$$\frac{du}{dx} = \frac{1}{x} \text{ and } \frac{dv}{dx} = 3 \cos 3x$$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\frac{\sin 3x}{x} - 3 \ln x \cos 3x}{\sin^2 3x}$$

$$= \frac{\sin 3x - 3x \ln x \cos 3x}{x \sin^2 3x}$$

When $x = \frac{\pi}{9}$,

$$\frac{dy}{dx} = \frac{\sin \frac{\pi}{3} - \frac{\pi}{3} \ln \left(\frac{\pi}{9} \right) \cos \frac{\pi}{3}}{\frac{\pi}{9} \sin^2 \frac{\pi}{3}}$$

$$= \frac{\frac{\sqrt{3}}{2} - \frac{\pi}{6} \ln \left(\frac{\pi}{9} \right)}{\frac{3\pi}{36}} = \frac{18\sqrt{3} - 6\pi \ln \left(\frac{\pi}{9} \right)}{3\pi}$$

8 a $x = \frac{e^y}{3+2y}$

When $y = 0$, $x = \frac{e^0}{3} = \frac{1}{3}$

Coordinates of P are $\left(\frac{1}{3}, 0\right)$.

b Let $u = e^y$ and $v = 3+2y$

$\frac{du}{dy} = e^y$ and $\frac{dv}{dy} = 2$

$\frac{dx}{dy} = \frac{e^y(3+2y) - 2e^y}{(3+2y)^2} = \frac{e^y(2y+1)}{(3+2y)^2}$

Gradient of normal to the curve is

$-\frac{1}{\frac{dx}{dy}} = -\frac{dy}{dx} = -\frac{e^y(2y+1)}{(3+2y)^2}$

Gradient of normal at $P\left(\frac{1}{3}, 0\right)$ is

$-\frac{e^0(2 \times 0 + 1)}{3^2} = -\frac{1}{9}$

Equation of normal at P is

$y - 0 = -\frac{1}{9}\left(x - \frac{1}{3}\right)$

$y = -\frac{1}{9}x + \frac{1}{27}$

This is in the form $y = mx + c$ with

$m = -\frac{1}{9}$ and $c = \frac{1}{27}$

9 Let $y = \frac{x^4}{\cos 3x}$

Let $u = x^4$ and $v = \cos 3x$

$\frac{du}{dx} = 4x^3$ and $\frac{dv}{dx} = -3\sin 3x$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{4x^3 \cos 3x - x^4(-3\sin 3x)}{\cos^2 3x}$
 $= \frac{x^3(4\cos 3x + 3x\sin 3x)}{\cos^2 3x}$

10 a $y = \frac{e^{2x}}{(x-2)^2}$

Let $u = e^{2x}$ and $v = (x-2)^2$

$\frac{du}{dx} = 2e^{2x}$ and $\frac{dv}{dx} = 2(x-2)$

Using $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{2(x-2)^2 e^{2x} - 2e^{2x}(x-2)}{(x-2)^4}$

$= \frac{2e^{2x}(x-2)((x-2)-1)}{(x-2)^4}$

$= \frac{2e^{2x}(x-3)}{(x-2)^3}$

So $A = 2$, $B = 1$ and $C = 3$.

b When $x = 1$, $y = e^2$

and $\frac{dy}{dx} = \frac{2e^2(-2)}{-1} = 4e^2$

Equation of tangent is

$y - e^2 = 4e^2(x - 1)$

$y = 4e^2x - 3e^2$

11 a $f(x) = \frac{2x}{x+5} + \frac{6x}{x^2+7x+10}$

$f(x) = \frac{2x}{x+5} + \frac{6x}{(x+2)(x+5)}$

$= \frac{2x(x+2)}{(x+2)(x+5)} + \frac{6x}{(x+2)(x+5)}$

$= \frac{2x^2 + 4x + 6x}{(x+2)(x+5)} = \frac{2x^2 + 10x}{(x+2)(x+5)}$

$= \frac{2x(x+5)}{(x+2)(x+5)} = \frac{2x}{x+2}$

In the last line, dividing through by $(x+5)$ is allowed because $x > 0$ so $x+5 \neq 0$.

b Let $u = 2x$ and $v = x+2$

$\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 1$

$f'(x) = \frac{2(x+2) - 2x}{(x+2)^2} = \frac{4}{(x+2)^2}$

Hence $f'(3) = \frac{4}{5^2} = \frac{4}{25}$

$$12 \text{ a } f(x) = \frac{2 \cos 2x}{e^{2-x}}$$

Let $u = 2 \cos 2x$ and $v = e^{2-x}$

$$\frac{du}{dx} = -4 \sin 2x \text{ and } \frac{dv}{dx} = -e^{2-x}$$

$$\begin{aligned} f'(x) &= \frac{-4e^{2-x} \sin 2x - 2 \cos 2x (-e^{2-x})}{(e^{2-x})^2} \\ &= \frac{2e^{2-x}(\cos 2x - 2 \sin 2x)}{(e^{2-x})^2} \end{aligned}$$

At stationary points, $f'(x) = 0$

$$\cos 2x - 2 \sin 2x = 0$$

$$2 \sin 2x = \cos 2x$$

$$\therefore \tan 2x = \frac{1}{2}$$

- b** The range of $f(x)$ is between the y -coordinate of B and the y -coordinate of the right endpoint of the interval.

$$\tan 2x = \frac{1}{2} \Rightarrow 2x = 0.4636 \text{ or } 3.6052$$

$$x = 0.2318 \text{ or } 1.8026$$

So the x -coordinate of B is 1.8026.

Range of $f(x)$ is

$$f(1.8026) \leq y < f(\pi)$$

$$-1.47 \leq y < 6.26 \text{ (3 s.f.)}$$