

Trigonometry and modelling 7D

1 a $3\cos q = 2\sin(q + 60^\circ)$

$$\Rightarrow 3\cos \theta = 2(\sin \theta \cos 60^\circ + \cos \theta \sin 60^\circ)$$

$$\Rightarrow 3\cos \theta = 2\left(\frac{1}{2}\sin \theta + \frac{\sqrt{3}}{2}\cos \theta\right) = \sin \theta + \sqrt{3}\cos \theta$$

$$\Rightarrow (3 - \sqrt{3})\cos \theta = \sin \theta$$

$$\Rightarrow \tan \theta = 3 - \sqrt{3} = 1.2679\dots \quad \left(\text{as } \tan \theta = \frac{\sin \theta}{\cos \theta}\right)$$

As $\tan \theta$ is positive, θ is in the first and third quadrants

$$\theta = \tan^{-1}(1.2679), 180^\circ + \tan^{-1}(1.2679)$$

$$\theta = 51.7^\circ, 231.7^\circ$$

b $\sin(q + 30^\circ) + 2\sin q = 0$

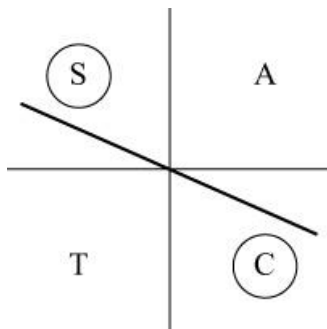
$$\Rightarrow \sin \theta \cos 30^\circ + \cos \theta \sin 30^\circ + 2\sin \theta = 0$$

$$\Rightarrow \frac{\sqrt{3}}{2}\sin \theta + \frac{1}{2}\cos \theta + 2\sin \theta = 0$$

$$\Rightarrow (4 + \sqrt{3})\sin \theta = -\cos \theta$$

$$\Rightarrow \tan \theta = -\frac{1}{4 + \sqrt{3}}$$

As $\tan \theta$ is negative, θ is in the second and fourth quadrants



$$\theta = \tan^{-1}\left(-\frac{1}{4 + \sqrt{3}}\right) + 180^\circ, \tan^{-1}\left(-\frac{1}{4 + \sqrt{3}}\right) + 360^\circ$$

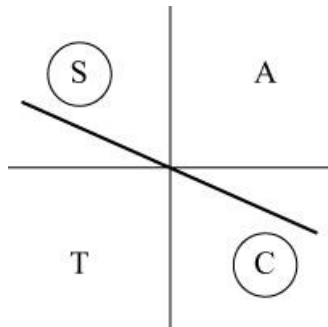
$$\theta = 170.1^\circ, 350.1^\circ$$

1 c $\cos(q + 25^\circ) + \sin(q + 65^\circ) = 1$
 $\Rightarrow \cos \theta \cos 25^\circ - \sin \theta \sin 25^\circ + \sin \theta \cos 65^\circ + \cos \theta \sin 65^\circ = 1$
 As $\sin(90 - x)^\circ = \cos x^\circ$ and $\cos(90 - x)^\circ = \sin x^\circ$
 $\cos 25^\circ = \sin 65^\circ$ and $\sin 25^\circ = \cos 65^\circ$
 So $\cos \theta \sin 65^\circ - \sin \theta \cos 65^\circ + \sin \theta \cos 65^\circ + \cos \theta \sin 65^\circ = 1$
 $\Rightarrow 2 \cos \theta \sin 65^\circ = 1$
 $\Rightarrow \cos \theta = \frac{1}{2 \sin 65^\circ} = 0.55168\dots$

$\theta = \cos^{-1}(0.55168), 360^\circ - \cos^{-1}(0.55168)$
 $\theta = 56.5^\circ, 303.5^\circ$

d $\cos q = \cos(q + 60^\circ)$
 $\Rightarrow \cos \theta = \cos \theta \cos 60^\circ - \sin \theta \sin 60^\circ = \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta$
 $\Rightarrow \cos \theta = -\sqrt{3} \sin \theta$
 $\Rightarrow \tan \theta = -\frac{1}{\sqrt{3}} \quad \left(\text{as } \tan \theta = \frac{\sin \theta}{\cos \theta} \right)$

As $\tan \theta$ is negative, θ is in the second and fourth quadrants



$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + 180^\circ, \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + 360^\circ$
 $\theta = 150.0^\circ, 330.0^\circ$

2 a $\sin\left(\theta + \frac{\pi}{4}\right) \equiv \sin \theta \cos \frac{\pi}{4} + \cos \theta \sin \frac{\pi}{4}$
 $\equiv \frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta \equiv \frac{1}{\sqrt{2}} (\sin \theta + \cos \theta)$

$$2 \text{ b } \frac{1}{\sqrt{2}}(\sin \theta + \cos \theta) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Find all answers for $\theta + \frac{\pi}{4}$. As $0 \leq \theta \leq 2\pi$ so $\frac{\pi}{4} \leq \theta + \frac{\pi}{4} \leq 2\pi + \frac{\pi}{4}$

$$\theta + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \text{ so } \theta = 0, \frac{\pi}{2}, 2\pi$$

$$c \text{ As } \sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\sin \theta + \cos \theta)$$

When $\sin \theta + \cos \theta = 1$

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\text{So } \theta = 0, \frac{\pi}{2}, 2\pi$$

$$3 \text{ a } \cos \theta \cos 30^\circ - \sin \theta \sin 30^\circ = 0.5$$

$$\Rightarrow \cos(\theta + 30^\circ) = 0.5$$

$$\Rightarrow \theta + 30^\circ = 60^\circ, 300^\circ$$

$$\Rightarrow \theta = 30^\circ, 270^\circ$$

$$b \text{ } \cos \theta \cos 30^\circ - \sin \theta \sin 30^\circ \equiv \cos \theta \times \frac{\sqrt{3}}{2} - \sin \theta \times \frac{1}{2}$$

So $\cos \theta \cos 30^\circ - \sin \theta \sin 30^\circ = \frac{1}{2}$ is identical to $\sqrt{3} \cos \theta - \sin \theta = 1$

Solutions are same as (a), i.e. $30^\circ, 270^\circ$

$$4 \text{ a } 3\sin(x - y) - \sin(x + y) = 0$$

$$\Rightarrow 3\sin x \cos y - 3\cos x \sin y - \sin x \cos y - \cos x \sin y = 0$$

$$\Rightarrow 2\sin x \cos y = 4\cos x \sin y$$

$$\Rightarrow \frac{2\sin x \cos y}{\cos x \cos y} = \frac{4\cos x \sin y}{\cos x \cos y}$$

$$\Rightarrow \frac{2\sin x}{\cos x} = \frac{4\sin y}{\cos y}$$

$$\Rightarrow 2 \tan x = 4 \tan y$$

$$b \text{ Put } y = 45^\circ \quad \triangleright \quad \tan x = 2$$

$$\text{So } x = \tan^{-1} 2, 180^\circ + \tan^{-1} 2$$

$$x = 63.4^\circ, 243.4^\circ \text{ (1 d.p.)}$$

$$\begin{aligned}
 \mathbf{5\ a} \quad \sin 2\theta &= \sin \theta, \quad 0 \leq \theta \leq 2\pi \\
 &\Rightarrow 2 \sin \theta \cos \theta = \sin \theta \\
 &\Rightarrow 2 \sin \theta \cos \theta - \sin \theta = 0 \\
 &\Rightarrow \sin \theta (2 \cos \theta - 1) = 0 \\
 &\Rightarrow \sin \theta = 0 \text{ or } \cos \theta = \frac{1}{2}
 \end{aligned}$$

$$\text{Solution set: } 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$$

$$\begin{aligned}
 \mathbf{b} \quad \cos 2\theta &= 1 - \cos \theta, \quad -180^\circ < \theta \leq 180^\circ \\
 &\Rightarrow 2 \cos^2 \theta - 1 = 1 - \cos \theta \\
 &\Rightarrow 2 \cos^2 \theta + \cos \theta - 2 = 0 \\
 &\Rightarrow \cos \theta = \frac{-1 \pm \sqrt{17}}{4} \quad (\text{using the quadratic formula})
 \end{aligned}$$

$$\text{As } \frac{-1 - \sqrt{17}}{4} \leq -1, \text{ this gives the only solution as } \cos \theta = \frac{-1 + \sqrt{17}}{4} = 0.78077\dots$$

As $\cos \theta$ is positive, θ is in the first and fourth quadrants

Using a calculator $\cos^{-1} 0.78077 = 38.7^\circ$ (1 d.p.)

Solutions are $\pm 38.7^\circ$

$$\begin{aligned}
 \mathbf{c} \quad 3 \cos 2\theta &= 2 \cos^2 \theta, \quad 0 \leq \theta < 360^\circ \\
 &\Rightarrow 3(2 \cos^2 \theta - 1) = 2 \cos^2 \theta \\
 &\Rightarrow 6 \cos^2 \theta - 3 = 2 \cos^2 \theta \\
 &\Rightarrow 4 \cos^2 \theta = 3 \\
 &\Rightarrow \cos^2 \theta = \frac{3}{4} \Rightarrow \cos \theta = \pm \frac{\sqrt{3}}{2}
 \end{aligned}$$

θ will be in all four quadrants.

Solution set: $30^\circ, 150^\circ, 210^\circ, 330^\circ$

$$\begin{aligned}
 \mathbf{d} \quad \sin 4\theta &= \cos 2\theta, \quad 0 \leq \theta \leq \pi \\
 &\Rightarrow 2 \sin 2\theta \cos 2\theta = \cos 2\theta \\
 &\Rightarrow \cos 2\theta (2 \sin 2\theta - 1) = 0 \\
 &\Rightarrow \cos 2\theta = 0 \text{ or } \sin 2\theta = \frac{1}{2}
 \end{aligned}$$

$\cos 2\theta = 0$ in $0 \leq 2\theta \leq 2\pi$

$$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$\sin 2\theta = \frac{1}{2}$ in $0 \leq 2\theta \leq 2\pi$

$$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

Solution set: $\frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$

5 e $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0, \quad 0 \leq \theta \leq 720^\circ$

$$\Rightarrow 3\left(1 - 2\sin^2\frac{\theta}{2}\right) - \sin\frac{\theta}{2} - 1 = 0$$

$$\Rightarrow 6\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} - 2 = 0$$

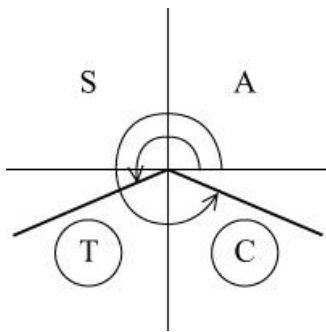
$$\Rightarrow \left(3\sin\frac{\theta}{2} + 2\right)\left(2\sin\frac{\theta}{2} - 1\right) = 0$$

$$\Rightarrow \sin\frac{\theta}{2} = -\frac{2}{3} \text{ or } \sin\frac{\theta}{2} = \frac{1}{2}$$

$$\sin\frac{\theta}{2} = \frac{1}{2} \text{ in } 0 \leq \frac{\theta}{2} \leq 360^\circ$$

$$\Rightarrow \frac{\theta}{2} = 30^\circ, 150^\circ \Rightarrow \theta = 60^\circ, 300^\circ$$

$$\sin\frac{\theta}{2} = -\frac{2}{3} \text{ in } 0 \leq \frac{\theta}{2} \leq 360^\circ$$



$$\Rightarrow \frac{\theta}{2} = 180^\circ - \sin^{-1}\left(-\frac{2}{3}\right), \quad 360^\circ + \sin^{-1}\left(-\frac{2}{3}\right) = 221.8^\circ, 318.2^\circ \text{ (1 d.p.)}$$

$$\Rightarrow \theta = 443.6^\circ, 636.4^\circ$$

Solution set: $60^\circ, 300^\circ, 443.6^\circ, 636.4^\circ$

f $\cos^2\theta - \sin 2\theta = \sin^2\theta, \quad 0 \leq \theta \leq \pi$

$$\Rightarrow \cos^2\theta - \sin^2\theta = \sin 2\theta$$

$$\Rightarrow \cos 2\theta = \sin 2\theta$$

$$\Rightarrow \tan 2\theta = 1 \quad (\text{divide both sides by } \cos 2\theta)$$

$$\tan 2\theta = 1 \text{ in } 0 \leq 2\theta \leq 2\pi$$

$$\Rightarrow 2\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\Rightarrow \theta = \frac{\pi}{8}, \frac{5\pi}{8}$$

5 g $2\sin\theta = \sec\theta, \quad 0 \leq \theta \leq 2\pi$

$$\Rightarrow 2\sin\theta = \frac{1}{\cos\theta}$$

$$\Rightarrow 2\sin\theta\cos\theta = 1$$

$$\Rightarrow \sin 2\theta = 1$$

$$\sin 2\theta = 1 \text{ in } 0 \leq 2\theta \leq 4\pi$$

$$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

h $2\sin 2q = 3\tan q, \quad 0 \leq q < 360^\circ$

$$\Rightarrow 4\sin\theta\cos\theta = \frac{3\sin\theta}{\cos\theta}$$

$$\Rightarrow 4\sin\theta\cos^2\theta = 3\sin\theta$$

$$\Rightarrow \sin\theta(4\cos^2\theta - 3) = 0$$

$$\Rightarrow \sin\theta = 0 \text{ or } \cos^2\theta = \frac{3}{4}$$

$$\sin\theta = 0 \Rightarrow \theta = 0^\circ, 180^\circ$$

$$\cos^2\theta = \frac{3}{4} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

$$\text{Solution set: } 0^\circ, 30^\circ, 150^\circ, 180^\circ, 210^\circ, 330^\circ$$

5 i $2 \tan \theta = \sqrt{3}(1 - \tan \theta)(1 + \tan \theta), \quad 0 \leq \theta \leq 2\pi$

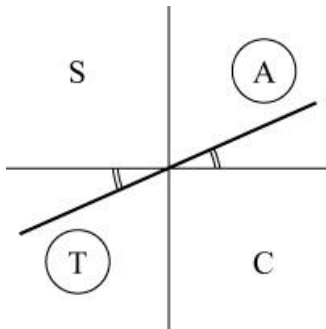
$$\Rightarrow 2 \tan \theta = \sqrt{3}(1 - \tan^2 \theta)$$

$$\Rightarrow \sqrt{3} \tan^2 \theta + 2 \tan \theta - \sqrt{3} = 0$$

$$\Rightarrow (\sqrt{3} \tan \theta - 1)(\tan \theta + \sqrt{3}) = 0$$

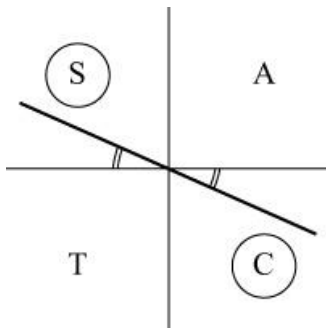
$$\Rightarrow \tan \theta = \frac{1}{\sqrt{3}} \text{ or } \tan \theta = -\sqrt{3}$$

$$\tan \theta = \frac{1}{\sqrt{3}}, \quad 0 \leq \theta \leq 2\pi$$



$$\Rightarrow \theta = \tan^{-1} \frac{1}{\sqrt{3}}, \quad \pi + \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \frac{7\pi}{6}$$

$$\tan \theta = -\sqrt{3}, \quad 0 \leq \theta \leq 2\pi$$



$$\Rightarrow \theta = \pi + \tan^{-1}(-\sqrt{3}), \quad 2\pi + \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}, \quad \frac{5\pi}{3}$$

Solution set: $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

5 j $\sin^2 q = 2\sin 2q, \quad -180^\circ < q \leq 180^\circ$

$$\Rightarrow \sin^2 \theta = 4\sin \theta \cos \theta$$

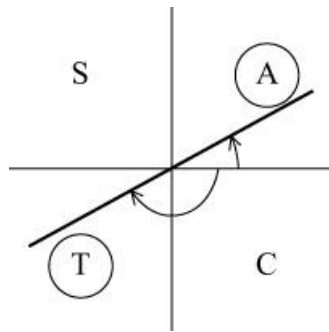
$$\Rightarrow \sin \theta(\sin \theta - 4\cos \theta) = 0$$

$$\Rightarrow \sin \theta = 0 \text{ or } \sin \theta = 4\cos \theta$$

$$\Rightarrow \sin \theta = 0 \text{ or } \tan \theta = 4$$

$$\sin \theta = 0 \Rightarrow \theta = 0^\circ, 180^\circ$$

$$\tan \theta = 4 \Rightarrow \theta = \tan^{-1} 4, -180^\circ + \tan^{-1} 4 = 76.0^\circ, -104.0^\circ \text{ (1 d.p.)}$$



Solution set: $-104.0^\circ, 0^\circ, 76.0^\circ, 180^\circ$

k $4\tan q = \tan 2q, \quad 0 \leq q < 360^\circ$

$$\Rightarrow 4\tan \theta = \frac{2\tan \theta}{1 - \tan^2 \theta}$$

$$\Rightarrow 2\tan \theta(1 - \tan^2 \theta) = \tan \theta$$

$$\Rightarrow \tan \theta(2 - 2\tan^2 \theta - 1) = 0$$

$$\Rightarrow \tan \theta(1 - 2\tan^2 \theta) = 0$$

$$\Rightarrow \tan \theta = 0 \text{ or } \tan \theta = \pm\sqrt{\frac{1}{2}}$$

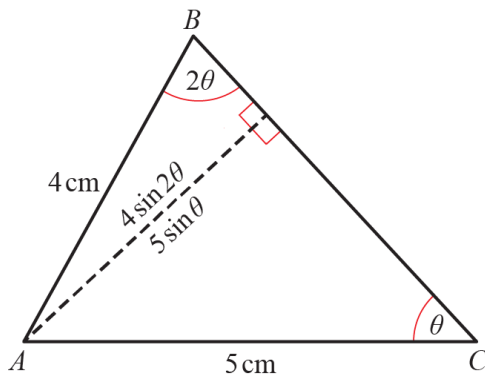
$$\tan \theta = 0 \Rightarrow \theta = 0^\circ, 180^\circ$$

$$\tan \theta = \pm\sqrt{\frac{1}{2}}$$

$$\Rightarrow \theta = 35.3^\circ, 144.7^\circ, 215.3^\circ, 324.7^\circ$$

Solution set: $0^\circ, 35.3^\circ, 144.7^\circ, 180^\circ, 215.3^\circ, 324.7^\circ$

6 Sketch $\triangle ABC$



$$4 \sin 2\theta = 5 \sin \theta$$

$$\Rightarrow 8 \sin \theta \cos \theta = 5 \sin \theta$$

$$\Rightarrow 8 \sin \theta \cos \theta - 5 \sin \theta = 0$$

$$\Rightarrow \sin \theta (8 \cos \theta - 5) = 0$$

$$\Rightarrow \sin \theta = 0 \text{ or } \cos \theta = \frac{5}{8}$$

As ABC is a triangle, $0 < \theta < 90^\circ$, so $\theta = 0^\circ$ or 180° are not possible solutions.

$$\text{So } \theta = \cos^{-1}\left(\frac{5}{8}\right) = 51.3^\circ \text{ (1 d.p.)}$$

7 a As $5 \sin 2\theta = 10 \sin \theta \cos \theta$

$$5 \sin 2\theta + 4 \sin \theta = 10 \sin \theta \cos \theta + 4 \sin \theta = 0$$

$$2 \sin \theta (5 \cos \theta + 2) = 0$$

$$\text{So } a = 2, b = 5 \text{ and } c = 2$$

b $2 \sin \theta (5 \cos \theta + 2) = 0, 0 \leq \theta < 360^\circ$

$$\Rightarrow \sin \theta = 0 \text{ or } \cos \theta = -\frac{2}{5}$$

$$\sin \theta = 0 \Rightarrow \theta = 0^\circ, 180^\circ$$

$$\cos \theta = -\frac{2}{5} \Rightarrow \theta = \cos^{-1}\left(-\frac{2}{5}\right), 360^\circ - \cos^{-1}\left(-\frac{2}{5}\right) = 113.6^\circ, 246.4^\circ \text{ (1 d.p.)}$$

$$\text{Solution set: } \theta = 0^\circ, 113.6^\circ, 180^\circ, 246.4^\circ$$

8 a As $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = 1 - 2 \sin^2 \theta$

$$\sin 2\theta + \cos 2\theta = 1$$

$$\Rightarrow 2 \sin \theta \cos \theta + (1 - 2 \sin^2 \theta) = 1$$

$$\Rightarrow 2 \sin \theta \cos \theta - 2 \sin^2 \theta = 0$$

$$\Rightarrow 2 \sin \theta (\cos \theta - \sin \theta) = 0$$

8 b $2\sin\theta(\cos\theta - \sin\theta) = 0, \quad 0 \leq \theta < 360^\circ$

$$\Rightarrow \sin\theta = 0 \text{ or } \cos\theta = \sin\theta$$

$$\sin\theta = 0 \Rightarrow \theta = 0^\circ, 180^\circ$$

$$\cos\theta = \sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^\circ, 225^\circ$$

$$\text{Solution set: } \theta = 0^\circ, 45^\circ, 180^\circ, 225^\circ$$

9 a LHS $\equiv (\cos 2\theta - \sin 2\theta)^2$

$$\equiv \cos^2 2\theta - 2\sin 2\theta \cos 2\theta + \sin^2 2\theta$$

$$\equiv (\cos^2 2\theta + \sin^2 2\theta) - (2\sin 2\theta \cos 2\theta)$$

$$\equiv 1 - \sin 4\theta \quad (\sin^2 A + \cos^2 A \equiv 1, \sin 2A \equiv 2\sin A \cos A)$$

$$\equiv \text{RHS}$$

b You can use $(\cos 2\theta - \sin 2\theta)^2 = \frac{1}{2}$ but this also solves the equation

$$\cos 2\theta - \sin 2\theta = -\frac{1}{\sqrt{2}}$$

so you need to check your final answers.

$$\text{As } (\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$$

$$\Rightarrow \frac{1}{2} = 1 - \sin 4\theta$$

$$\Rightarrow \sin 4\theta = \frac{1}{2}$$

$$0 \leq \theta < \pi, \text{ so } 0 \leq 4\theta < 4\pi$$

$$\Rightarrow 4\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}$$

Checking these values in $\cos 2\theta - \sin 2\theta = \frac{1}{\sqrt{2}}$ eliminates $\frac{5\pi}{24}, \frac{13\pi}{24}$

which apply to $\cos 2\theta - \sin 2\theta = -\frac{1}{\sqrt{2}}$

Solutions are $\frac{\pi}{24}, \frac{17\pi}{24}$

$$\begin{aligned}
 \mathbf{10\ a\ i}\quad \text{RHS} &\equiv \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \\
 &\equiv \frac{2 \tan \frac{\theta}{2}}{\sec^2 \frac{\theta}{2}} \\
 &\equiv \frac{2 \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \times \cos^2 \frac{\theta}{2} \\
 &\equiv 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\
 &\equiv \sin \theta \quad (\sin 2A = 2 \sin A \cos A) \\
 &\equiv \text{LHS}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii}\quad \text{RHS} &\equiv \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \\
 &\equiv \frac{1 - \tan^2 \frac{\theta}{2}}{\sec^2 \frac{\theta}{2}} \\
 &\equiv \cos^2 \frac{\theta}{2} \left(1 - \tan^2 \frac{\theta}{2}\right) \\
 &\equiv \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \quad \left(\tan^2 \frac{\theta}{2} = \frac{\sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}}\right) \\
 &\equiv \cos \theta \quad (\cos 2A = \cos^2 A - \sin^2 A) \\
 &\equiv \text{LHS}
 \end{aligned}$$

$$\mathbf{b}\quad \text{Let } \tan \frac{\theta}{2} = t$$

$$\begin{aligned}
 \mathbf{i}\quad \sin \theta + 2 \cos \theta &= 1 \\
 \Rightarrow \frac{2t}{1+t^2} + \frac{2(1-t^2)}{1+t^2} &= 1 \\
 \Rightarrow 2t + 2 - 2t^2 &= 1 + t^2 \\
 \Rightarrow 3t^2 - 2t - 1 &= 0 \\
 \Rightarrow (3t+1)(t-1) &= 0 \\
 \Rightarrow \tan \frac{\theta}{2} = -\frac{1}{3} \text{ or } \tan \frac{\theta}{2} = 1, \quad 0 \leq \frac{\theta}{2} \leq 180^\circ \\
 \tan \frac{\theta}{2} = 1 \Rightarrow \frac{\theta}{2} = 45^\circ \Rightarrow \theta &= 90^\circ \\
 \tan \frac{\theta}{2} = -\frac{1}{3} \Rightarrow \frac{\theta}{2} = 161.56^\circ \Rightarrow \theta &= 323.1^\circ \text{ (1 d.p.)} \\
 \text{Solution set: } &90^\circ, 323.1^\circ
 \end{aligned}$$

$$10 \text{ b ii } 3\cos\theta - 4\sin\theta = 2$$

$$\Rightarrow \frac{3(1-t^2)}{1+t^2} - \frac{4 \times 2t}{1+t^2} = 2$$

$$\Rightarrow 3(1-t^2) - 8t = 2(1+t^2)$$

$$\Rightarrow 5t^2 + 8t - 1 = 0$$

$$\Rightarrow t = \frac{-8 \pm \sqrt{84}}{10}$$

$$\text{For } \tan \frac{\theta}{2} = \frac{-8 + \sqrt{84}}{10} \quad 0 \leq \frac{\theta}{2} \leq 180^\circ$$

$$\frac{\theta}{2} = 6.65^\circ \Rightarrow \theta = 13.3^\circ \text{ (1 d.p.)}$$

$$\text{For } \tan \frac{\theta}{2} = \frac{-8 - \sqrt{84}}{10} \quad 0 \leq \frac{\theta}{2} \leq 180^\circ$$

$$\frac{\theta}{2} = 120.2^\circ \Rightarrow \theta = 240.4^\circ \text{ (1 d.p.)}$$

Solution set: $13.3^\circ, 240.4^\circ$

$$11 \text{ a RHS } \equiv 1 + 2\cos 2x$$

$$\equiv 1 + 2(\cos^2 x - \sin^2 x)$$

$$\equiv 1 + 2\cos^2 x - 2\sin^2 x$$

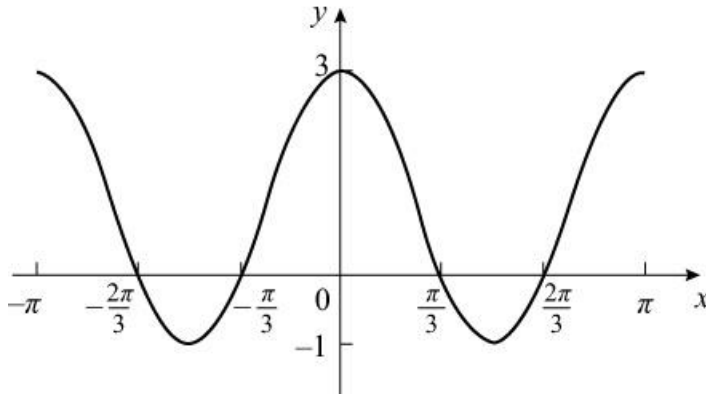
$$\equiv \cos^2 x + \sin^2 x + 2\cos^2 x - 2\sin^2 x \quad (\text{using } \sin^2 x + \cos^2 x \equiv 1)$$

$$\equiv 3\cos^2 x - \sin^2 x$$

$$\equiv \text{LHS}$$

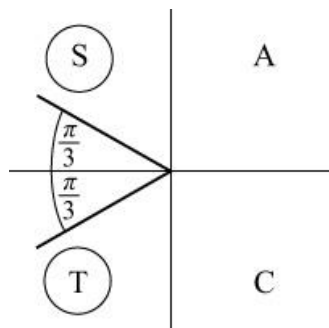
11 b $y = 3\cos^2 x - \sin^2 x$ is the same as $y = 1 + 2\cos 2x$

Using your work on transformations, this curve is the result of
 (i) stretching $y = \cos x$ by scale factor $\frac{1}{2}$ in the x direction, then
 (ii) stretching the result by scale factor 2 in the y direction, then
 (iii) translating by 1 in the positive y direction.



The curve crosses y -axis at $(0, 3)$. It crosses x -axis where $y = 0$
 i.e. where $1 + 2\cos 2x = 0 \quad -\pi \leq x \leq \pi$

$$\Rightarrow \cos 2x = -\frac{1}{2} \quad -2\pi \leq 2x \leq 2\pi$$



$$\text{So } 2x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\Rightarrow x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$$

The curve meets the x -axis at $\left(-\frac{2\pi}{3}, 0\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{2\pi}{3}, 0\right)$

12 a $\cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}, \quad \sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$

$$\text{So } 2\cos^2 \frac{\theta}{2} - 4\sin^2 \frac{\theta}{2} = (1 + \cos \theta) - 2(1 - \cos \theta) = 3\cos \theta - 1$$

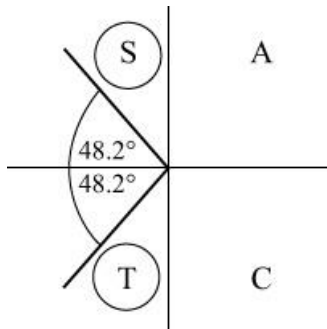
12 b $3\cos\theta - 1 = -3, \quad 0 \leq \theta < 360^\circ$

$$\Rightarrow 3\cos\theta = -2$$

$$\Rightarrow \cos\theta = -\frac{2}{3}$$

As $\cos\theta$ is negative, θ is in second and third quadrants.

Calculator value is $\cos^{-1}\left(-\frac{2}{3}\right) = 131.8^\circ$ (1 d.p.)



Solutions are $131.8^\circ, 360^\circ - 131.8^\circ = 228.2^\circ$ (1 d.p.)

13 a As $\sin^2 A + \cos^2 A \equiv 1$ so $(\sin^2 A + \cos^2 A)^2 \equiv 1$

$$\Rightarrow \sin^4 A + \cos^4 A + 2\sin^2 A \cos^2 A \equiv 1$$

$$\Rightarrow \sin^4 A + \cos^4 A \equiv 1 - 2\sin^2 A \cos^2 A$$

$$\equiv 1 - \frac{1}{2}(4\sin^2 A \cos^2 A)$$

$$\equiv 1 - \frac{1}{2}((2\sin A \cos A)^2)$$

$$\equiv 1 - \frac{1}{2}\sin^2 2A$$

$$\equiv \frac{1}{2}(2 - \sin^2 2A)$$

b As $\cos 2A \equiv 1 - 2\sin^2 A$ so $\cos 4A \equiv 1 - 2\sin^2 2A$ so $\sin^2 2A \equiv \frac{1 - \cos 4A}{2}$

$$\Rightarrow \text{from (a) } \sin^4 A + \cos^4 A \equiv \frac{1}{2}\left(2 - \frac{1 - \cos 4A}{2}\right) \equiv \frac{1}{2}\left(\frac{4 - 1 + \cos 4A}{2}\right) \equiv \frac{1}{4}(3 + \cos 4A)$$

13 c Using part (b)

$$8\sin^4 \theta + 8\cos^4 \theta = 7$$

$$\Rightarrow 8 \times \frac{1}{4}(3 + \cos 4\theta) = 7$$

$$\Rightarrow 3 + \cos 4\theta = \frac{7}{2}$$

$$\Rightarrow \cos 4\theta = \frac{1}{2}$$

$$\text{Solve } \cos 4\theta = \frac{1}{2} \text{ in } 0 < 4\theta < 4\pi$$

$$\Rightarrow 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

14 a $\cos 3\theta \equiv \cos(2\theta + \theta) \equiv \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$
 $\equiv (\cos^2 \theta - \sin^2 \theta) \cos \theta - (2 \sin \theta \cos \theta) \sin \theta$
 $\equiv \cos^3 \theta - \sin^2 \theta \cos \theta - 2 \sin^2 \theta \cos \theta$
 $\equiv \cos^3 \theta - 3 \sin^2 \theta \cos \theta$
 $\equiv \cos^3 \theta - 3(1 - \cos^2 \theta) \cos \theta$
 $\equiv 4 \cos^3 \theta - 3 \cos \theta$

b $6 \cos \theta - 8 \cos^3 \theta + 1 = 0, \quad 0 < \theta < \pi$

$$\Rightarrow 1 = 8 \cos^3 \theta - 6 \cos \theta$$

$$\Rightarrow 4 \cos^3 \theta - 3 \cos \theta = \frac{1}{2}$$

$$\Rightarrow \cos 3\theta = \frac{1}{2}, \quad 0 < 3\theta < 3\pi \quad \text{using the result from part (a)}$$

$$\text{So } 3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$