Trigonometry and modelling 7A

1 a i $\angle FAB = \angle CAF + \angle BAC$
$= (\alpha - \beta) + \beta = \alpha$
So $\angle FAB = \alpha$

ii $\angle FAB$ and $\angle ABD$ are alternate angles
so $\angle FAB = \angle ABD$
so $\angle ABD = \alpha$
$\angle CBE = 90 - \alpha$, so $\angle ECB = 90 - (90 - \alpha) = \alpha$

iii $\cos\beta = \frac{AB}{1}$
So $AB = \cos\beta$

iv $\sin\beta = \frac{BC}{1}$
So $BC = \sin\beta$

b i $\angle ABD = \alpha$, so $\sin\alpha = \frac{AD}{AB}$

As $AB = \cos\beta$, this gives $\sin\alpha = \frac{AD}{\cos\beta}$

So $AD = \sin\alpha\cos\beta$

ii $\cos\alpha = \frac{BD}{AB} = \frac{BD}{\cos\beta}$
So $BD = \cos\alpha\cos\beta$

c i $\angle ECB = \alpha$, so $\cos\alpha = \frac{CE}{BC}$

As $BC = \sin\beta$, this gives $\cos\alpha = \frac{CE}{\sin\beta}$

So $CE = \cos\alpha\sin\beta$

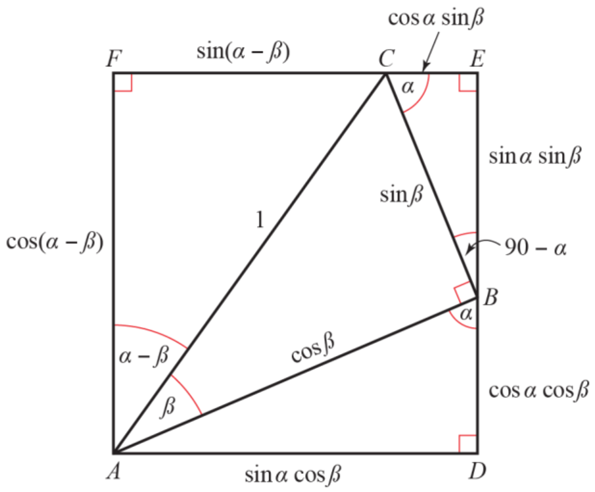
ii $\sin\alpha = \frac{BE}{BC} = \frac{BE}{\sin\beta}$
So $BE = \sin\alpha\sin\beta$

d i $\sin(\alpha - \beta) = \frac{FC}{1}$
So $FC = \sin(\alpha - \beta)$

1 d ii $\cos(\alpha-\beta)=\dfrac{FA}{1}$

So $FA=\cos(\alpha-\beta)$

e i The completed diagram should look like this:



$FC + CE = AD$, so $FC = AD - CE$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

ii AF = DB + BE

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

2 $$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}$$
$$=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

Divide the numerator and denominator by cos A cos B

$$\tan(A-B)=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}+\dfrac{\sin A\sin B}{\cos A\cos B}}=\frac{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A-\tan B}{1+\tan A\tan B}$$ as required

3 $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\sin(P+(-Q))=\sin P\cos(-Q)+\cos P\sin(-Q)$

As $\cos(-P)=\cos P$ and $\sin(-P)=-\sin P$, this gives

$\sin(P-Q)\equiv\sin P\cos Q-\cos P\sin Q$

4 Example: $A = 60°,\ B = 30°$

$$\sin A = \frac{\sqrt{3}}{2},\ \sin B = \frac{1}{2}$$

$$\sin(A+B) = 1;\ \sin A + \sin B = \frac{\sqrt{3}}{2} + \frac{1}{2} \neq 1$$

This proves $\sin(A+B) = \sin A + \sin B$ is not true for all values.

There will be many values of A and B for which the statement is true, e.g. $A = -30°$ and $B = +30°$, and this is the danger of trying to prove a statement by taking particular examples. To prove a statement requires a sound argument; to disprove it only requires one counterexample.

5 $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

Set $A = \theta,\ B = \theta$

$$\Rightarrow \quad \cos(\theta - \theta) \equiv \cos\theta\cos\theta + \sin\theta\sin\theta$$

$$\Rightarrow \quad \cos 0 \equiv \cos^2\theta + \sin^2\theta$$

So $\cos^2\theta + \sin^2\theta \equiv 1$ (since $\cos 0 = 1$)

6 **a** $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

Set $A = \frac{\pi}{2},\ B = \theta$

$$\Rightarrow \quad \sin\left(\frac{\pi}{2} - \theta\right) \equiv \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{2} - \theta\right) \equiv \cos\theta$$

since $\sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0$

b $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

Set $A = \frac{\pi}{2},\ B = \theta$

$$\Rightarrow \quad \cos\left(\frac{\pi}{2} - \theta\right) \equiv \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta$$

since $\cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1$

7 $\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}$

$$= \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$

8 $\cos\left(x+\frac{\pi}{3}\right)=\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}$

$=\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$

9 a Using $\sin(A+B)\equiv\sin A\cos B+\cos A\sin B$ gives

$\sin 15°\cos 20°+\cos 15°\sin 20°\equiv\sin(15°+20°)\equiv\sin 35°$

b Using $\sin(A-B)\equiv\sin A\cos B-\cos A\sin B$ gives

$\sin 58°\cos 23°-\cos 58°\sin 23°\equiv\sin(58°-23°)\equiv\sin 35°$

c Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives

$\cos 130°\cos 80°-\sin 130°\sin 80°\equiv\cos(130°+80°)\equiv\cos 210°$

d Using $\tan(A-B)\equiv\frac{\tan A-\tan B}{1+\tan A\tan B}$ gives

$\frac{\tan 76°-\tan 45°}{1+\tan 76°\tan 45°}\equiv\tan(76°-45°)\equiv\tan 31°$

e Using $\cos(A-B)\equiv\cos A\cos B+\sin A\sin B$ gives

$\cos 2\theta\cos\theta+\sin 2\theta\sin\theta\equiv\cos(2\theta-\theta)\equiv\cos\theta$

f Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives

$\cos 4\theta\cos 3\theta-\sin 4\theta\sin 3\theta\equiv\cos(4\theta+3\theta)\equiv\cos 7\theta$

g Using $\sin(A+B)\equiv\sin A\cos B+\cos A\sin B$ gives

$\sin\tfrac{1}{2}\theta\cos 2\tfrac{1}{2}\theta+\cos\tfrac{1}{2}\theta\sin 2\tfrac{1}{2}\theta\equiv\sin\left(\tfrac{1}{2}\theta+2\tfrac{1}{2}\theta\right)\equiv\sin 3\theta$

h Using $\tan(A+B)\equiv\frac{\tan A+\tan B}{1-\tan A\tan B}$ gives

$\frac{\tan 2\theta+\tan 3\theta}{1-\tan 2\theta\tan 3\theta}\equiv\tan(2\theta+3\theta)\equiv\tan 5\theta$

i Using $\sin(P-Q)\equiv\sin P\cos Q-\cos P\sin Q$ gives

$\sin(A+B)\cos B-\cos(A+B)\sin B\equiv\sin\left((A+B)-B\right)\equiv\sin A$

j Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives

$\cos\left(\frac{3x+2y}{2}\right)\cos\left(\frac{3x-2y}{2}\right)-\sin\left(\frac{3x+2y}{2}\right)\sin\left(\frac{3x-2y}{2}\right)\equiv\cos\left(\left(\frac{3x+2y}{2}\right)+\left(\frac{3x-2y}{2}\right)\right)$

$\equiv\cos\left(\frac{6x}{2}\right)\equiv\cos 3x$

10 a Use the fact that $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\sin x + \cos x) = \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4} = \sin\left(x + \frac{\pi}{4}\right)$$

or

$$\frac{1}{\sqrt{2}}(\sin x + \cos x) = \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4} = \cos\left(x - \frac{\pi}{4}\right)$$

b Use the fact that $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\cos x - \sin x) == \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \cos x\cos\frac{\pi}{4} - \sin x\sin\frac{\pi}{4} = \cos\left(x + \frac{\pi}{4}\right)$$

c Use the fact that $\frac{1}{2} = \cos\frac{\pi}{3} = \sin\frac{\pi}{6}$ and $\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} = \sin\frac{\pi}{3}$ to write

$$\frac{1}{2}(\sin x + \sqrt{3}\cos x) = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3} = \sin\left(x + \frac{\pi}{3}\right)$$

or

$$\frac{1}{2}(\sin x + \sqrt{3}\cos x) == \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \cos\left(x - \frac{\pi}{6}\right)$$

d Use the fact that $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\sin x - \cos x) = \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x = \sin x\cos\frac{\pi}{4} - \cos x\sin\frac{\pi}{4} = \sin\left(x - \frac{\pi}{4}\right)$$

11 $\cos y = \sin(x + y)$

$\Rightarrow \cos y = \sin x\cos y + \cos x\sin y$

Divide throughout by $\cos x\cos y$

$$\frac{\cancel{\cos y}^{1}}{\cos x\cancel{\cos y}} = \frac{\sin x\cancel{\cos y}}{\cos x\cancel{\cos y}} + \frac{\cancel{\cos x}\sin y}{\cancel{\cos x}\cos y}$$

Þ $\sec x = \tan x + \tan y$

Þ $\tan y = \sec x -\ \tan x$

12 As $\tan(x - y) = 3$

so $\frac{\tan x - \tan y}{1 + \tan x\tan y} = 3$

$\Rightarrow \tan x - \tan y = 3 + 3\tan x\tan y$

$\Rightarrow 3\tan x\tan y + \tan y = \tan x - 3$

$\Rightarrow \tan y(3\tan x + 1) = \tan x - 3$

$\Rightarrow \tan y = \frac{\tan x - 3}{3\tan x + 1}$

13 $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$

$\Rightarrow \sin x\cos y + 2\sin x\sin y = 2\cos x\cos y - \cos x\sin y$

$\Rightarrow \sin x\cos y + \cos x\sin y = 2(\cos x\cos y - \sin x\sin y)$

$\Rightarrow \sin(x+y) = 2\cos(x+y)$

$\Rightarrow \dfrac{\sin(x+y)}{\cos(x+y)} = 2$

$\Rightarrow \tan(x+y) = 2$

14 a $\tan(x - 45°) = \dfrac{1}{4}$

$\Rightarrow \dfrac{\tan x - \tan 45°}{1 + \tan x\tan 45°} = \dfrac{1}{4}$

$\Rightarrow 4\tan x - 4 = 1 + \tan x$ (as $\tan 45° = 1$)

$\Rightarrow 3\tan x = 5$

$\Rightarrow \tan x = \dfrac{5}{3}$

b $\sin(x - 60°) = 3\cos(x + 30°)$

$\Rightarrow \sin x\cos 60° - \cos x\sin 60° = 3\cos x\cos 30° - 3\sin x\sin 30°$

$\Rightarrow \dfrac{1}{2}\sin x - \dfrac{\sqrt{3}}{2}\cos x = \dfrac{3\sqrt{3}}{2}\cos x - \dfrac{3}{2}\sin x$

$\Rightarrow 4\sin x = 4\sqrt{3}\cos x$

$\Rightarrow \dfrac{\sin x}{\cos x} = \dfrac{4\sqrt{3}}{4}$

$\Rightarrow \tan x = \sqrt{3}$

c $\tan(x - 60°) = 2$

$\Rightarrow \dfrac{\tan x - \tan 60°}{1 + \tan x\tan 60°} = 2$

$\Rightarrow \dfrac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x} = 2$ (as $\tan 60° = \sqrt{3}$)

$\Rightarrow \tan x - \sqrt{3} = 2 + 2\sqrt{3}\tan x$

$\Rightarrow (2\sqrt{3} - 1)\tan x = -(2 + \sqrt{3})$

$\Rightarrow \tan x = -\dfrac{(2+\sqrt{3})}{2\sqrt{3}-1} = -\dfrac{(2+\sqrt{3})(2\sqrt{3}+1)}{(2\sqrt{3}-1)(2\sqrt{3}+1)}$

$= -\dfrac{8 + 5\sqrt{3}}{11}$

15 $\tan\left(x+\frac{\pi}{3}\right)=\frac{1}{2}$

$$\Rightarrow \frac{\tan x+\tan\frac{\pi}{3}}{1-\tan x\tan\frac{\pi}{3}}=\frac{1}{2}$$

$$\Rightarrow \frac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}=\frac{1}{2} \quad \left(\tan\frac{\pi}{3}=\sqrt{3}\right)$$

$$\Rightarrow 2\tan x+2\sqrt{3}=1-\sqrt{3}\tan x$$

$$\Rightarrow (2+\sqrt{3})\tan x=1-2\sqrt{3}$$

$$\Rightarrow \tan x=\frac{1-2\sqrt{3}}{2+\sqrt{3}}=\frac{(1-2\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$$

$$=\frac{2-4\sqrt{3}-\sqrt{3}+6}{1}=8-5\sqrt{3}$$

16 Write $\theta=\left(\theta+\frac{2\pi}{3}\right)-\frac{2\pi}{3}$ and $\theta+\frac{4\pi}{3}=\left(\theta+\frac{2\pi}{3}\right)+\frac{2\pi}{3}$

Now use the appropriate addition formulae for cos

$$\cos\left(\left(\theta+\frac{2\pi}{3}\right)-\frac{2\pi}{3}\right)=\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\sin\left(\theta+\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}$$

$$\cos\left(\left(\theta+\frac{2\pi}{3}\right)+\frac{2\pi}{3}\right)=\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}-\sin\left(\theta+\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}$$

Now add up all terms

$$\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$\equiv\cos\left(\left(\theta+\frac{2\pi}{3}\right)-\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\left(\theta+\frac{2\pi}{3}\right)+\frac{2\pi}{3}\right)$$

$$\equiv\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\sin\left(\theta+\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}$$

$$-\sin\left(\theta+\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}$$

$$\equiv 2\cos\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\cos\left(\theta+\frac{2\pi}{3}\right)$$

$$\equiv 0 \text{ as } \cos\frac{2\pi}{3}=-\frac{1}{2}$$

Challenge

a i Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}x(y\cos B)(\sin A)$

$= \frac{1}{2}xy\sin A\cos B$

ii Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}y(x\cos A)(\sin B)$

$= \frac{1}{2}xy\cos A\sin B$

iii Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}xy\sin(A+B)$

b Area $T_1 + T_2 =$ Area $T_1 +$ Area T_2

$\Rightarrow \frac{1}{2}xy\sin(A+B) = \frac{1}{2}xy\sin A\cos B + \frac{1}{2}xy\cos A\sin B$

$\Rightarrow \sin(A+B) = \sin A\cos B + \cos A\sin B$