Integration 11E

1 a $\int x\sqrt{1+x}\ \mathrm{d}x$

Let $u = 1+x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

So $\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

$$\text{So } I = \int (u-1)u^{\frac{1}{2}}\ \mathrm{d}u$$

$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) \mathrm{d}u$$

$$= \frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} + c$$

$$= \frac{2(1+x)^{\frac{5}{2}}}{5} - \frac{2(1+x)^{\frac{3}{2}}}{3} + c$$

b $\int \frac{1+\sin x}{\cos x}\ \mathrm{d}x$

Let $u = \sin x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$

So $\mathrm{d}x$ can be replaced by $\frac{\mathrm{d}u}{\cos x}$.

$$\text{So } I = \int \frac{1+u}{\cos^2 x}\ \mathrm{d}u$$

$$= \int \frac{1+u}{1-\sin^2 x}\ \mathrm{d}u$$

$$= \int \frac{1+u}{1-u^2}\ \mathrm{d}u$$

$$= \int \frac{1}{1-u}\ \mathrm{d}u$$

$$= -\ln|1-u| + c$$

$$= -\ln|1-\sin x| + c$$

c $\int \sin^3 x\ \mathrm{d}x$

Let $u = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$

So $\sin x\mathrm{d}x$ can be replaced by $-\mathrm{d}u$.

Now $\sin^3 x = \sin x(1-\cos^2 x)$,

$$\text{so } I = \int \sin x(1-\cos^2 x)\ \mathrm{d}x$$

$$= \int (u^2-1)\ \mathrm{d}u$$

$$= \frac{u^3}{3} - u + c$$

$$= \frac{\cos^3 x}{3} - \cos x + c$$

d $\int \frac{2}{\sqrt{x}(x-4)}\ \mathrm{d}x$

Let $u = \sqrt{x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$

So $\frac{\mathrm{d}x}{\sqrt{x}}$ can be replaced by $2\mathrm{d}u$.

$$\text{So } I = \int \frac{4}{(u^2-4)}\ \mathrm{d}u$$

$$= \int \frac{4}{(u-2)(u+2)}\ \mathrm{d}u$$

$$= \int \left(\frac{1}{u-2} - \frac{1}{u+2}\right) \mathrm{d}u$$

$$= \ln|u-2| - \ln|u+2| + c$$

$$= \ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + c$$

1 e $\int \sec^2 x \tan x\sqrt{1+\tan x}\ \mathrm{d}x$

Let $u^2 = 1+\tan x$

$2u\frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$

So $\sec^2 x\mathrm{d}x$ can be replaced by $2u\mathrm{d}u$.

So $I = \int 2u(u^2-1)u\ \mathrm{d}u$

$= \int\left(2u^4 - 2u^2\right)\ \mathrm{d}u$

$= \frac{2u^5}{5} - \frac{2u^3}{3} + c$

$= \frac{2(1+\tan x)^{\frac{5}{2}}}{5} - \frac{2(1+\tan x)^{\frac{3}{2}}}{3} + c$

f $\int \sec^4 x\ \mathrm{d}x$

Let $u = \tan x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$

So $\sec^2 x\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

Now $\sec^2 x = 1+\tan^2 x$

So $I = \int(1+\tan^2 x)\sec^2 x\ \mathrm{d}x$

$= \int(1+u^2)\ \mathrm{d}u$

$= u + \frac{u^3}{3} + c$

$= \tan x + \frac{\tan^3 x}{3} + c$

2 a $\int_0^5 x\sqrt{x+4}\ \mathrm{d}x$

Let $u = x+4$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

So $\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

x	u
5	9
0	4

So $I = \int_4^9 (u-4)\sqrt{u}\ \mathrm{d}u$

$= \int_4^9 (u^{\frac{3}{2}} - 4u^{\frac{1}{2}})\ \mathrm{d}u$

$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right]_4^9$

$= \left(\frac{486}{5} - \frac{216}{3}\right) - \left(\frac{64}{5} - \frac{64}{3}\right)$

$= \frac{506}{15}$

b $\int_0^2 x(2+x)^3\ \mathrm{d}x$

Let $u = 2+x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

So $\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

x	u
2	4
0	2

So $I = \int_2^4 (u-2)u^3\ \mathrm{d}u$

$= \left[\frac{u^5}{5} - \frac{u^4}{2}\right]_2^4$

$= \left(\frac{1024}{5} - \frac{256}{2}\right) - \left(\frac{32}{5} - \frac{16}{2}\right)$

$= \frac{768}{10} + \frac{16}{10} = \frac{784}{10} = \frac{392}{5}$

2 c $\int_0^{\frac{\pi}{2}} \sin x\sqrt{3\cos x+1}\,\mathrm{d}x$

Let $u=\cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=-\sin x$

So $\sin x\mathrm{d}x$ can be replaced by $-\mathrm{d}u$.

x	u
$\frac{\pi}{2}$	0
0	1

So $I=\int_1^0 -(3u+1)^{\frac{1}{2}}\,\mathrm{d}u$

Consider $y=(3u+1)^{\frac{3}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{9}{2}(3u+1)^{\frac{1}{2}}$

$I=\left[-\frac{2}{9}(3u+1)^{\frac{3}{2}}\right]_1^0$

$=-\frac{2}{9}+\frac{16}{9}=\frac{14}{9}$

d $\int_0^{\frac{\pi}{3}} \sec x\tan x\sqrt{\sec x+2}\,\mathrm{d}x$

Let $u=\sec x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\sec x\tan x$

So $\sec x\tan x\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

x	u
$\frac{\pi}{3}$	2
0	1

So $I=\int_1^2 (u+2)^{\frac{1}{2}}\,\mathrm{d}u$

$=\left[\frac{2}{3}(u+2)^{\frac{3}{2}}\right]_1^2$

$=\frac{16}{3}-2\sqrt{3}$

2 e $\int_1^4 \frac{1}{\sqrt{x}(4x-1)}\,\mathrm{d}x$

Let $u=\sqrt{x}$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}$

So $\frac{\mathrm{d}x}{\sqrt{x}}$ can be replaced by $2\mathrm{d}u$.

x	u
4	2
1	1

So $I=\int_1^2 \frac{2}{(4u^2-1)}\,\mathrm{d}u$

$=\int_1^2\left(\frac{1}{2u-1}-\frac{1}{2u+1}\right)\mathrm{d}u$

$=\left[\frac{1}{2}\ln|2u-1|-\frac{1}{2}\ln|2u+1|\right]_1^2$

$=\frac{1}{2}\ln 3-\frac{1}{2}\ln 5-\frac{1}{2}\ln 1+\frac{1}{2}\ln 3$

$=\ln 3-\frac{1}{2}\ln 5$

$=\frac{1}{2}\ln 9-\frac{1}{2}\ln 5$

$=\frac{1}{2}\ln\frac{9}{5}$

3 a $\int x(3+2x)^5\,\mathrm{d}x$

Let $u=3+2x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=2$

So $2\mathrm{d}x$ can be replaced by $\mathrm{d}u$.

$I=\int\frac{(u-3)}{4}u^5\,\mathrm{d}u$

$=\int\frac{u^6-3u^5}{4}\,\mathrm{d}u$

$=\frac{u^7}{28}-\frac{3u^6}{24}+c$

$=\frac{(3+2x)^7}{28}-\frac{(3+2x)^6}{8}+c$

3 b $\int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x$

Let $u = 1+x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

So dx can be replaced by du.

$I = \int \frac{(u-1)}{u^{\frac{1}{2}}}\,\mathrm{d}u$

$I = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)\mathrm{d}u$

$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + c$

$= \frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + c$

c $\int \frac{\sqrt{x^2+4}}{x}\,\mathrm{d}x$

Let $u = \sqrt{x^2+4}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = x(x^2+4)^{-\frac{1}{2}} = \frac{x}{u}$

$I = \int \frac{u}{x} \times \frac{u}{x}\,\mathrm{d}u$

$= \int \frac{u^2}{x^2}\,\mathrm{d}u = \int \frac{u^2}{u^2-4}\,\mathrm{d}u$

$= \int \left(1 + \frac{1}{u-2} - \frac{1}{u+2}\right)\mathrm{d}u$

$= u + \left(\ln|u-2| - \ln|u+2|\right) + c$

$= \sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2}\right| + c$

4 a $\int_2^7 x\sqrt{2+x}\,\mathrm{d}x$

Let $u = 2+x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

So dx can be replaced by du.

x	u
7	9
2	4

So $I = \int_4^9 (u-2)\sqrt{u}\,\mathrm{d}u$

$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}}\right]_4^9$

$= \left(\frac{486}{5} - \frac{108}{3}\right) - \left(\frac{64}{5} - \frac{32}{3}\right)$

$= \frac{886}{15}$

b $\int_2^5 \frac{1}{1+\sqrt{x-1}}\,\mathrm{d}x$

Let $u = \sqrt{x-1}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x-1}} = \frac{1}{2u}$

So dx can be replaced by $2u$du.

x	u
5	2
2	1

So $I = \int_1^2 \frac{2u}{1+u}\,\mathrm{d}u$

$I = \int_1^2 2\left(1 - \frac{1}{1+u}\right)\mathrm{d}u$

$= \left[2u - 2\ln|1+u|\right]_1^2$

$= (4 - 2\ln 3) - (2 - 2\ln 2)$

$= 2 + 2\ln 2 - 2\ln 3$

$= 2 + 2\ln\frac{2}{3}$

4 c $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\, d\theta$

Let $u = 1+\cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

So $d\theta$ can be replaced by $\frac{du}{\sin\theta}$.

θ	u
$\frac{\pi}{2}$	1
1	2

So $I = -\int_2^1 \frac{2\sin\theta\cos\theta}{u}\ \frac{du}{\sin\theta}$

$= \int_2^1 \frac{2(1-u)}{u}\ du$

$= \left[2\ln|u| - 2u\right]_2^1$

$= -2-(2\ln 2-4)$

$= 2-2\ln 2$

5 $\int_6^{20} \frac{8x}{\sqrt{4x+1}}\ dx$

Let $u^2 = 4x+1$

$2u\frac{du}{dx} = 4$

So dx can be replaced by $\frac{u}{2}du$.

x	u
20	9
6	5

So $I = \int_5^9 \frac{2(u^2-1)}{u}\ \frac{u}{2}du$

$= \left[\frac{u^3}{3} - u\right]_5^9$

$= \left(\frac{792}{3}-9\right) - \left(\frac{125}{3}-5\right)$

$= \frac{592}{3}$

6 $\int_{\ln 3}^{\ln 4} \frac{e^{4x}}{e^x-2}\ dx$

Let $u^2 = e^x - 2$

$2u\frac{du}{dx} = e^x = u^2+2$

So dx can be replaced by $\frac{2u}{u^2+2}du$.

x	u
ln 4	$\sqrt{2}$
ln 3	1

$I = \int_1^{\sqrt{2}} \frac{2(u^3+2)^3}{u}\ du$

$I = \int_1^{\sqrt{2}} \left(2u^5 + 12u^3 + 24u + \frac{16}{u}\right)\ du$

$= \left[\frac{1}{3}u^6 + 3u^4 + 12u^2 + 16\ln|u|\right]_1^{\sqrt{2}}$

$= \frac{70}{3} + 8\ln 2$

$a = 70, b = 3, c = 8, d = 2$

7 $-\int \frac{1}{\sqrt{1-x^2}}\ dx$

Let $x = \cos\theta$

$\frac{dx}{d\theta} = -\sin\theta$

So dx can be replaced by $-\sin\theta\ d\theta$.

$\sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$

$I = -\int \frac{1}{\sin\theta}(-\sin\theta)\ d\theta = \int 1\ d\theta$

$= \theta + c$

$= \arccos x + c$

8 $\int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x \, \mathrm{d}x$

Let $u = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$

So $\mathrm{d}x$ can be replaced by $-\frac{\mathrm{d}u}{\sin x}$.

x	u
$\frac{\pi}{3}$	$\frac{1}{2}$
0	1

$I = \int_1^{\frac{1}{2}} -(1-u^2)u^2 \, \mathrm{d}u$

$I = \int_1^{\frac{1}{2}} (u^2 - 1)u^2 \, \mathrm{d}u$

$= \int_1^{\frac{1}{2}} (u^4 - u^2) \, \mathrm{d}u$

$= \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^{\frac{1}{2}}$

$= \left(\frac{1}{160} - \frac{1}{24}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)$

$= \frac{47}{480}$

9 $I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2 \sqrt{1-x^2} \, \mathrm{d}x$

Let $x = \sin\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta$

x	θ
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
$\frac{1}{2}$	$\frac{\pi}{6}$

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta \cos^2\theta \, \mathrm{d}\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{4}\sin^2 2\theta \, \mathrm{d}\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{8}(1 - \cos 4\theta) \, \mathrm{d}\theta$

$= \left[\frac{1}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{1}{8}\left(\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)\right)$

$= \frac{1}{8}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$

$= \frac{2\pi + 3\sqrt{3}}{96}$

Challenge

$$\int \frac{1}{x^2\sqrt{9-x^2}}\,\mathrm{d}x$$

Let $x = 3\sin u$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 3\cos u$$

So $\mathrm{d}x$ can be replaced by $3\cos u\ \mathrm{d}u$.

$$\int \frac{3\cos u}{9\sin^2 u\sqrt{9-9\sin^2 u}}\,\mathrm{d}u$$

$$= \int \frac{3\cos u}{9\sin^2 u 3\cos u}\,\mathrm{d}u$$

$$= \frac{1}{9}\int \operatorname{cosec}^2 u\ \mathrm{d}u$$

$$= -\frac{1}{9}\cot u + c$$

$$I = -\frac{\cos u}{9\sin u} + c$$

$$\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-\frac{x^2}{9}} = \frac{\sqrt{9-x^2}}{3}$$

$$I = -\frac{\frac{\sqrt{9-x^2}}{3}}{\frac{9x}{3}} + c$$

$$= -\frac{\sqrt{9-x^2}}{9x} + c$$